Ph.D Qualify Exam **Differential Equations** **2022 Spring**

E: Easy; M: Moderate; D: Difficult

1(E, 20%, 2006, Fall). Use the Fourier transform method to solve the initial value problem

$$\begin{aligned} &u_t = u_{xx}, && -\infty < x < \infty,\ t > 0; \\ &u(x,0) = f(x), && -\infty < x < \infty. \end{aligned}$$

And prove that the solution u satisfies the inequality

$$\|u\|_p(t) \leqslant \frac{1}{(4\pi t)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}} \|f\|_q, \quad t > 0,$$

for $1 \le q \le p \le \infty$. (Note that the L^p, L^q norms are respect to x.)

2(E, 20%, 2006, Fall). Let $u(r,\theta)$ be a harmonic function in the disk

$$D = \{(r,\theta) \mid 0 \le r < R, -\pi < \theta \le \pi\},$$

such that u is continuous in the closed disk $\overline{D}$ and satisfies

$$u(R,\theta) = \begin{cases} \sin^2(2\theta), & |\theta| \le \pi/2, \\ 0, & \pi/2 < \theta \leqslant \pi. \end{cases}$$

(a) Evaluate $u(0,0)$.
(b) Show that $0 < u(r,\theta) < 1$ holds at each point (r,θ) in the disk.

3(E, 20%, 2001, Fall). Let u be a solution of the wave equation in all of $\mathbb{R}^3 \times \mathbb{R}$. Suppose that $a > 0$ and that $u(x,0) = u_t(x,0) = 0$ for $|x| \ge a$.
(a) Show that $u(x,t) = 0$ in the double cone $|x| \le |t| - a$ for $|t| \ge a$.
(b) Show that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^3} u^2(x,t)dx \le C$$

for all $t > 0$.

4(M, 20%, 2017, Fall). Consider the heat equation $u_t(x,t) = u_{xx}(x,t)$ with the boundary condition

$$\begin{cases} u(x,0) = U_1, & x < 0, \\ u(x,0) = U_2, & x > 0, \end{cases}$$

where U_1 and U_2 are constants. (1) Please solve the problem. Is the solution unique? (2) Please find $\lim\limits_{t\to\infty} u(x,t)$ for each $x \in \mathbb{R}$.

5(E, 20%, 2007, Fall). Let $f \in C^1(\mathbb{R}^n)$ and suppose that for each open ball B, there exists a solution of the bounded value problem

$$\begin{cases} -\Delta u = f & \text{in } B \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases}$$

where n is the outward unit normal vector field to ∂B. Show that $f \equiv 0$.