Qualifying Exam - Differential Geometry
Fall 2020

All manifolds in the problems below are smooth and finite dimensional, and all maps are smooth unless stated otherwise.

1. The space of matrices, $Mat(n, \mathbb{R})$ can be identified with $\mathbb{R}^{n^2}$ in the natural way. Define $GL(n, \mathbb{R})$ to be the set of *invertible* n-matrices, and $O(n, \mathbb{R})$ be the set of *orthogonal matrices.* Precisely,

$$GL(n, \mathbb{R}) := \{A \in Mat(n, \mathbb{R}) \mid det A \neq 0\},$$

$$O(n, \mathbb{R}) := \{A \in Mat(n, \mathbb{R}) \mid AA^T = I_n\}.$$

   (a) (5 points) (Easy) Prove that $GL(n, \mathbb{R})$ is a smooth manifold.

   (b) (5 points) (Very Easy) $O(n, \mathbb{R}) = F^{-1}(0)$ for some F from $GL(n, \mathbb{R})$ to the space of symmetric n-matrices. What is F?

   (c) (5 points) (Medium) Take for granted that $O(n, \mathbb{R})$ is a smooth submanifold of $GL(n, \mathbb{R})$ because the 0 matrix is a regular value for F in part (b). What is the dimension of $O(n, \mathbb{R})$?

   (d) (5 points) (Medium) Describe the tangent space $T_{I_n}O(n, \mathbb{R})$ completely with sufficient reasons.

2. Given $f : M \to \mathbb{R}$ and ω a p-form on M, prove

   (a) (5 points) (Easy)

$$d(f\omega) = df \wedge \omega.$$

   using local coordinates.

   (b) (10 points) (Easy) Use previous part to show that for a 1-form ω and two vector fields X, Y, we have

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

   Here, $[X, Y]$ is the Lie bracket of X and Y.

3. Given manifolds M, N of dimensions m and n, respectively, a smooth bijective map $f : M \to N$ so that df_p is injective everywhere,

   (a) (10 points) (Medium) Prove that $n = m$.

   (b) (10 points) (Easy) Prove that f is a diffeomorphism.

4. (10 points) (Easy) State the classical Green's Theorem for smooth vector field $V = (P, Q)$ on $\mathbb{R}^2$ and prove it using Stoke's theorem.

5. On an m-dimensional manifold M,

   (a) (5 points) (Easy) Define $H^p_{dR}(M)$, the p^{th} de Rham cohomology group.

   (b) (15 points) (Medium) Prove that $H^1_{dR}(U) = 0$ for *star-shaped* domain U in $\mathbb{R}^n$ with $n > 1$.

   Note: a domain U is star-shaped if there exists $a \in U$ so that every $x \in U$ can be joined with a by a line in U.

6. (15 points) (Medium) Given the fact that every vector field on $\mathbb{S}^2$ must vanishes somewhere, prove that $\mathbb{S}^2$ has no Lie group structure. (Hint: What is TG for a Lie group G in general?)