Ph.D Qualify Exam **Differential Equations** **2021 Fall**

E: Easy; M: Moderate; D: Difficult

1(E, 15%, 2003, Fall). Suppose that Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary. Let $u(x,t)$ be a smooth solution of the problem

$$\begin{cases} u_t - \Delta u + c(x,t)u = 0 & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial B \times (0,\infty) \\ u(x,0) = g(x) & \text{in } \Omega. \end{cases}$$

Suppose that $g(x) \geq 0$ for all $x \in \Omega$ and there exists a constant K such that $|c(x,t)| < K$ for all $(x,t) \in \overline{\Omega} \times (0,\infty)$. Show that $u(x,t) \geq 0$ for all $(x,t) \in \overline{\Omega} \times (0,\infty)$.

2(E, 10%, 2007, Fall). Let $f \in C^1(\mathbb{R}^n)$ and suppose that for each open ball B, there exists a solution of the bounded value problem

$$\begin{cases} -\Delta u = f & \text{in } B \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases}$$

where n is the outward unit normal vector field to ∂B. Show that $f \equiv 0$.

3(E, 15%, 2017, Fall). Let u be a nonconstant harmonic function in the disk $\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 < R^2\}$. Define for each $0 < r < R$,

$$M(r) = \max_{x^2+y^2=r^2} u(x,y).$$

Prove that M is a monotone increasing function in the interval $(0,R)$.

4(M, 15%, 2017, Fall). Consider the heat equation $u_t(x,t) = u_{xx}(x,t)$ with the boundary condition

$$\begin{cases} u(x,0) = U_1, & x < 0, \\ u(x,0) = U_2, & x > 0, \end{cases}$$

where U_1 and U_2 are constants. (1) Please solve the problem. Is the solution unique? (2) Please find $\lim_{t\to\infty} u(x,t)$ for each $x \in \mathbb{R}$.

5(E, 15%, 2018, Spring). Solve the linear Reaction-Diffusion-Advection equation

$$u_t = u + u_x + u_{xx}, \ x \in \mathbb{R}, \ t > 0,$$

with $u(x,0) = g(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

6(E, 15%, 2020, Spring). Using the energy method to prove the uniqueness of the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} + hu = F(x,t) & x \in \mathbb{R}, \ t > 0, \\ \lim_{x\to\pm\infty} u(x,t) = \lim_{x\to\pm\infty} u_x(x,t) = \lim_{x\to\pm\infty} u_t(x,t) = 0, & t > 0, \\ \int_{\mathbb{R}} u_t^2 + c^2 u_x^2 + hu^2 dx < \infty & t > 0, \\ u(x,0) = f(x), \ u_t(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$