# Ph.D Qualify Exam, Analysis, Apr. 1, 2020

Show all works

E: Easy, M: Moderate, D: Difficult

**1.**[10%] (Sept. 2008) Evaluate the integral $\int_{-\infty}^{\infty} e^{-x^2}\cos(xt)\,dx := I(t)$, $\lim_{t\to 0} I(t)$, and $\lim_{t\to\infty} I(t)$.

**2.**[10%] (Sept. 2014) Let $C$ be the Cantor set and $\varphi(x)$ be the Cantor function. **(a)** Show that the Lebesque measure of $C$ is zero. **(b)** Show that $C$ is uncountable. **(c)** Is $\varphi(x)$ continuous? uniformly continuous? absolutely continuous? Prove it.

**3.**[15%] (Sept. 2008) Let $\mathbf{T}(x,y) = (e^x\cos y - 1, e^x \sin y) = (u,v)$ be a transformation: $R^2 \to R^2$, and $f$ be a continuous function on $R^2$ with compact support. Let $J_{\mathbf{T}}$ be the Jacobian of $\mathbf{T}$. **(a)** Show that there are functions $g_1$ and $g_2$ from $R^2$ into $R^1$ such that $\mathbf{T}(x,y) = \mathbf{G_2} \circ \mathbf{G_1}(x,y)$, where $\mathbf{G_1}(x,y) = (g_1(x,y), y)$ and $\mathbf{G_2}(z,w) = (z, g_2(z,w))$. **(b)** Show that, for Riemann integral, $\int_{R^2} f(u,v)\,dudv = \int_{R^2} f(\mathbf{T}(x,y))\left|J_{\mathbf{T}}(x,y)\right|dxdy$. Use the result in part (a) to give a direct proof. **(c)** Under what conditions, does the formula in part (b) hold for Lebesgue integral?

**4.**[6%](Sept. 2004) Assume that $p > 0$ and $\int_E |f - f_k|^p dx \to 0$ as $k \to \infty$. Show that $\{f_k\}_{k=1}^{\infty}$ converges in measure on $E$ to $f$.

**5.**[20%](Feb. 2000 and Sept. 2014) Let $\mathcal{M}$ be the collection of Lebesgue measurable subsets of $R$. $\mu$ be the Lebesgue measure on $(R,\mathcal{M})$, and $\mu_0$ be the counting measure on $(R,\mathcal{M})$. Define $\nu$ on $(R,\mathcal{M})$ by $\nu(E) = \mu_0(E\cap\{0\}) - \mu(E\cap[0,1]) + \int_E \frac{1}{1+x^2}\,dx$. $(E \in \mathcal{M})$ **(a)** Find a Hahn decomposition of $R$ for measure $\nu$. **(b)** Find the Jordan decomposition of $\nu$. **(c)** Find the Lebesgue decomposition of $|\nu|$ with respect to $\mu$. **(d)** Compute the Radon-Nikodym derivative of the absolutely continuous part of $|\nu|$ with respect to $\mu$.

**6.**[15%] (Sept. 2014) **(a)** State the Lebesgue Domination Convergence Theorem.

**(b)** State the fundamental Theorems of Calculus for Riemann integral and for Lebesgue integral.

**(c)** State the Riesz representation theorem of the dual of $L^p(E)$.

**(d)** State a theorem for the dual of $\mathbb{R}^n$ in Linear Algebra, that is analogous with the Riesz representation theorem of the dual of $L^2(E)$. (Hint: Compare inner products in $L^2(E)$ and in $\mathbb{R}^n$)

**(e)** Let $f$ and $g$ be absolutely continuous. Show that the product rule holds.

**7.**[10%] (M) Prove the following theorem: Let $[a, b]$ be a closed, bounded interval and $1 \le p < \infty$. Suppose $T$ ia a bounded linear functional on $L^p[a,b]$. Then there is a function $g \in L^q[a,b]$, where $q$ is the conjugate of $p$, for which $T(f) = \int_a^b g\cdot f$ for all $f \in L^p[a,b]$. (Hint: Fundamental Theorem of Calculus for Lebesgue integral. Do not use Riesz representation theorem.)

**8.**[14%] (D) Let $E$ be a measurable set in $\mathbb{R}$ and $1 < p < \infty$. Assume that $\{f_n\} \rightharpoonup f$ weakly in $L^2(E)$ as $n \to \infty$. Show that

**(a)** There is a subsequence $\{f_{n_k}\}$ such that $\frac{1}{k}(f_{n_1} + \cdots + f_{n_k}) \to f$ strongly in $L^2(E)$ as $k \to \infty$.

**(b)** In addition, we suppose that $C$ is a closed, bounded convex subset of $L^2(E)$ and $T$ is a continuous convex functional on $C$. If $\{f_n\} \subset C$, then $f \in C$. Also we have $T(f) \le \liminf T(f_n)$.