PhD Qualify Exam: General Analysis

October 2019

E: Easy **M:Moderate** **D:Difficult**

1. (E, 10 points, 2017, 3) Let f be an absolutely continuous function on $[a, b] \subset \mathbb{R}$. If $Z \subset [a, b]$ is a Lebesgue measure zero set, then $f(Z)$ has Lebesgue measure zero.

2. (E, 15 points, 2018, 10) Suppose that f, $\{f_k\} \in L^p$ and that $f_k \to f$ a.e., $1 \leq p < \infty$. Show that $\|f_k - f\|_p \to 0$ iff $\|f_k\|_p \to \|f\|_p$.

3. (E, 15 points) Give an example of a sequence of measurable functions $\{f_k\}$ defined on a measurable set $E \subseteq \mathbb{R}^n$ such that the following strict inequalities hold:

$$\int_E \liminf_{k\to\infty} f_k dx < \liminf_{k\to\infty} \int_E f_k dx < \limsup_{k\to\infty} \int_E f_k dx < \int_E \limsup_{k\to\infty} f_k dx.$$

4. (E, 10 points) Give an example of a bounded function f defined on $(1, \infty)$ such that

$$f \in \bigcap_{p>1} L^p(1,\infty) \quad \text{but} \quad f \notin L^1(1,\infty).$$

5. (E, 15 points) Let $k(x, y)$ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^m$ satisfying that

$$\int_{\mathbb{R}^m} |k(x,y)|\, dy \leq C \quad \text{for a.e. } x,$$

$$\int_{\mathbb{R}^n} |k(x,y)|\, dx \leq C \quad \text{for a.e. } y,$$

where $C > 0$ is a univeral constant. Prove that

$$(Tf)(x) := \int_{\mathbb{R}^m} k(x,y) f(y)\, dy$$

is a bounded operator on $L^p(\mathbb{R}^n)$ with $\|Tf\|_p \leq C \|f\|_p$ for $1 \leq p \leq \infty$.

6. (M, 20 points, 2018, 3) Let $f : [0,1] \to [0, \infty)$ be a continuous function, show that

$$\int_0^1 f^2(x)\, dx \leq \frac{1000}{3} + \frac{1}{15\sqrt{10}} \int_0^1 f^3(x)\, dx.$$

7. (M, 15 points, 2019, 3) Suppose $1 < p < q < \infty$ and $p^{-1} + q^{-1} = 1$. If T is a bounded operator on L^p such that

$$\int (Tf)\, g = \int f\, (Tg)$$

for all f, $g \in L^p \cap L^q$, then T extends uniquely to a bounded operator on L^r for all $r \in [p, q]$.