Qualifying Examination in General Algebra

April 2020

- Attempt all problems. Show all your work and justify all your answers.
- Easier: 1, 2, 5; Medium: 3, 4, 6; Harder: 7

1. (10 points) Let p be a prime. Show that every group of order p^2 is abelian.

2. (10 points) Let I be a nonempty index set, and let A_i be a left R-module for all $i \in I$. Let B be a left R-module. Show that the groups $\mathrm{Hom}_R\left(\bigoplus_{i\in I} A_i, B\right)$ and $\prod_{i\in I} \mathrm{Hom}_R(A_i, B)$ are isomorphic. (*Here, for example,* $\mathrm{Hom}_R(A_i, B)$ *is the group of all R-module homomorphisms from A_i to B.*)

3. (15 points) Let n be a positive integer, and let R be a commutative ring with identity $1_R \neq 0$. Suppose F is a free R-module of rank n with basis $\{v_1, \ldots, v_n\}$ and A is a nonzero R-module. Show that every element of $A \otimes_R F$ can be written uniquely in the form $\sum_{i=1}^{n} x_i \otimes v_i$, where $x_1, \ldots, x_n \in A$.

4. (15 points) Let k, m, n be integers such that $k \geq 2$, $n \geq 2$, and $1 \leq m < n$. Show that if $V_1, \ldots, V_k$ are distinct real subspaces of $\mathbb{R}^n$ of dimension m, then $\bigcup_{i=1}^{k} V_i$ is not a real subspace of $\mathbb{R}^n$.

5. (15 points) Let E be the splitting field over $\mathbb{Q}$ of the polynomial $x^3 - 2 \in \mathbb{Q}[x]$. Determine all subfields of E.

6. (15 points) Let $\mathbb{Q}[x, y]$ be the polynomial ring in the variables x and y with coefficients in $\mathbb{Q}$. Determine if the rings $\mathbb{Q}[x, y]/(x^5 - y^{27})$ and $\mathbb{Q}[x, y]/(x^6 - y^{27})$ are isomorphic. *Justify your answer.*

7. (20 points) Let R be a unique factorization domain, and let S be a multiplicative subset of R such that $0 \notin S$. Show that $S^{-1}R$ is a unique factorization domain.