GENERAL ANALYSIS

(E: Easy, M:Moderate, D: Difficult) PhD Qualify Exam October 12, 2018

1. (M, 20 points, 2013, 3) Let (X, M, μ) be a measure space. Assume that $f \in L^r(X)$ for some $0 < r < \infty$. Show that
$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty .$$

2. (E, 15 points, 2017, 3) Let g_n and g be integrable functions, $g_n \to g$ a.e., and $|f_n| \leq g_n$, $f_n \to f$ a.e. If
$$\int g dx = \lim_{n \to \infty} \int g_n dx$$
then
$$\int f dx = \lim_{n \to \infty} \int f_n dx.$$

3. (E, 10 points) Let E be a subset of $\mathbb{R}$ with measure zero. Show that the set $\{x^2 : x \in E\}$ also has measure zero.

4. (E, 15 points) Suppose that f, $\{f_k\} \in L^p$ and that $f_k \to f$ a.e., $1 \leq p < \infty$. Show that $\|f_k - f\|_p \to 0$ iff $\|f_k\|_p \to \|f\|_p$.

5. (E, 15 points) Let f_k and f be (Lebesgue) measurable on a measurable set $E \subset \mathbb{R}^n$, $|E| < \infty$. Then
$$f_k \to f \quad \text{in measure} \quad \text{iff} \quad \int_E \frac{|f_k - f|}{1 + |f_k - f|} dx \to 0 \quad \text{as } k \to \infty.$$

6. (M, 15 points, 2015, 3) Let $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Prove that $f * g \in L^p(\mathbb{R}^n)$, and
$$\|f * g\|_p \leq \|f\|_p \|g\|_1 ,$$
where $(f * g)(x) = \int_{\mathbb{R}^n} f(t) g(x - t) dt$.

7. (E, 10 points) Suppose μ is a positive measure on X and $f : X \to (0, \infty)$ satisfies $\int_X f d\mu = 1$. Prove, for every $E \subset X$ with $0 < \mu(E) < \infty$, that
$$\int_E (\log f) \, d\mu \leq \mu(E) \log \frac{1}{\mu(E)}$$
and, when $0 < p < 1$,
$$\int_E f^p d\mu \leq \mu(E)^{1-p} .$$