Ph.D. QUALIFYING EXAM, GENERAL ANALYSIS

(E: easy; M: moderate; D: difficult)

1. **(E. 2017, #6. 15 points)** Let $E \subset \mathbb{R}^n$ be a measurable set with $|E| < \infty$. Define
$$N_p[f] := \left(\frac{1}{|E|}\int_E |f|^p\right)^{\frac{1}{p}}.$$
Prove that $N_p[f] \le N_q[f]$ for any $p \le q$.

2. **(E. 2018, #5. 15 points)** Prove the following statement: Suppose f_k, f are Lebesgue measurable on $E \subset \mathbb{R}^n$, $|E| < \infty$. Then
$$f_k \to f \text{ in measure if and only if } \int_E \frac{|f_k - f|}{1 + |f_k - f|} \to 0 \text{ as } k \to \infty.$$

3. **(M. 2016, #6. 20 points)** Let $1 \le p < \infty$ and g be an integrable function defined on $[0,1]$. Suppose that there exists $M > 0$ such that
$$\left|\int_{[0,1]} fg\right| \le M\|f\|_{L^p}$$
for all bounded measurable functions f. Please prove that $\|g\|_{L^q} \le M$ where $q = \frac{p}{p-1}$.

4. **(D. 4.3 Wheeden&Zygmund. 20 points)** Let $\{f_k\}$ be a sequence of measur -able functions defined on a measurable set E, $|E| < \infty$. If $|f_k(x)| \le M_x < \infty$ for all k for each $x \in E$, show that given $\varepsilon > 0$, there exists a closed $F \subset E$ and a finite M such that $|E - F| < \varepsilon$ and $|f_k(x)| \le M$ for all k and $x \in F$.

5. **(M. 6.5 Folland. 15 points)** Suppose $1 < p < q < \infty$ and $p^{-1} + q^{-1} = 1$. If T isa bounded operator on L^p such that
$$\int (Tf)g = \int f(Tg)$$
for all $f, g \in L^p \cap L^q$, then T extends uniquely to a bounded operator on L^r for all $r \in [p, q]$.

6. **(E. 4.5 Royden. 15 points)** A sequence $\{f_n\}$ of measurable function is called Cauchy sequence in measure if given $\varepsilon > 0$ there is N such that
$$m\{x \mid |f_n(x) - f_m(x)| \ge \varepsilon\} < \varepsilon$$
for all $m, n > N$. Show that $\{f_n\}$ converges in measure if $\{f_n\}$ is a Cauchy sequence in measure.

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