Algebra Exam Date: Friday 12/10/2018

Do all the following problems. Be sure to show all work and explain your reasoning as clearly as possible.

1. (a) (10%) Prove that for $n \geq 5$, the only normal subgroups of S_n are 1, A_n and S_n.

   (b) (10%) Prove that if p is a prime and P is a non-abelian group of order p^3 then $|Z(P)| = p$ and $P/Z(P) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$.

   (c) (10%) Prove that if $|G| = 1365$ then G is not simple.

2. (a) (10%) Prove that if $|G| = pq$ with p and q primes and $p < q$, then G is solvable and G has a normal subgroup of prime order.

   (b) (5%) Let p, q, r be three primes such that $p < q < r$ and G be a group with $|G| = pqr$. Prove that G is solvable.

3. (10%) Prove that a finite group G is nilpotent if and only if whenever $a, b \in G$ with $\gcd(|a|, |b|) = 1$ then $ab = ba$.

4. (a) (8%) An ideal N is called *nilpotent* if N^n is the zero ideal for some $n \geq 1$. Prove that the ideal $p\mathbb{Z}/p^m\mathbb{Z}$ is a nilpotent ideal in the ring $\mathbb{Z}/p^m\mathbb{Z}$.

   (b) (9%) Let K be a finite extension of F. Prove that K is a splitting field over F if and only if every irreducible polynomial in $F[x]$ that has a root in K splits completely in $K[x]$.

   (c) (8%) Let α and β be two algebraic elements over a field F. Assume that the degree of the minimal polynomial of α over F is relatively prime to the degree of the minimal polynomial of β over F. Prove that the minimal polynomial of β over F is irreducible over $F(\alpha)$.

5. (20%) Determine the Galois group of $(x^2 - 2)(x^2 - 3)(x^2 - 5)$. Determine all the subfields of the splitting field of this polynomial.