Qualifying Examination in General Algebra

March 2019

Attempt all 7 problems. Show all your work and justify all your answers.

1. (15 points) Let G be a finite group. Suppose p is the smallest prime dividing the order of G. Prove that every subgroup of G of index p is normal in G.

2. (15 points) Let p be an odd prime. Classify up to isomorphism all groups of order $2p$.

3. (15 points) Let R be a commutative ring with identity $1_R \neq 0$, and let I be an ideal of R. Prove that I is maximal if and only if R/I is a field.

4. (15 points) Let R be a commutative ring with identity $1_R \neq 0$, and let A be a free R-module of rank n with basis $\{\alpha_1, \ldots, \alpha_n\}$, where n is a positive integer. Suppose M is a nonzero R-module. Prove that every element of $M \otimes_R A$ can be written uniquely in the form $\sum_{i=1}^n m_i \otimes \alpha_i$, where $m_1, \ldots, m_n \in M$.

5. (15 points) For any prime p, let $\mathbb{F}_p$ be the finite field of order p. Find the number of monic irreducible quadratic polynomial in $\mathbb{F}_p[x]$.

6. (15 points) Let F be the splitting field over $\mathbb{Q}$ of the polynomial $x^7 - 1$, where $\mathbb{Q}$ is the field of rational numbers. Determine all subfields of F.

7. (10 points) Prove that every Artinian integral domain is a field.