(2017 Fall) **Qualifying Examination**
Subject: Mathematical Statistics

1(E, 15%). Let Z be a standard normal random variable and U be a chi-square random variable with n degree of freedom. Suppose that Z and U are independent. Please find the probability density function of $Z/\sqrt{U/n}$.

2(E, 15%). Let $x_1, ..., x_n$ be independently sampled from the Poisson distribution with parameter λ. (1) Find the asymptotic variance of the maximum likelihood estimate. (2) Find the method of moments estimate of λ.

3(M, 20%). Suppose that $\mu \sim N(2, \sigma_0^2)$ and that $X \mid \mu \sim N(\mu, \sigma^2)$. Derive a formula for the posterior mean of μ, given that n samples $x_1, ..., x_n$ of X are observed.

4(M, 20%). Let $X_1, ..., X_n$ be a sample from the standard normal distribution and let $Y_1, ..., Y_n$ be an independent sample from an $N(1, 1)$ distribution. Determine the mean and variance of the rank sum of $\{X_1, ..., X_n\}$.

5(M, 20%). Let X and Y be random variables with $E[X] = \mu_x$, $E[Y] = \mu_y$, $\text{Var}(X) = \sigma_x^2$, $\text{Var}(Y) = \sigma_y^2$, and $\text{Cov}(X, Y) = \sigma_{xy}$. Consider predicting Y from X as $Y = \alpha + \beta X$, where α and β are chosen to minimize the expected squared prediction error $E[(Y - Y)^2]$. Express α and β in terms of μ_x, μ_y, σ_x, σ_y, and σ_{xy}.

6(D, 10%). Let $X_1, ..., X_n$ be i.i.d. with density function $f(x \mid \theta)$, where f is differentiable. Let $T = t(X_1, ..., X_n)$ be an unbiased estimate of θ. Prove or disprove that $\text{Var}(T) \geq [nI(\theta)]^{-1}$, where $I(\theta) = \mathbb{E}\left[\frac{\partial}{\partial\theta} \ln f(x \mid \theta)\right]^2$.