Partial Differential Equation

Oct. 2017

E: easy, M: moderate, D: difficult.

1. (E, 10 points) Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

2. (E, 10 points) Let $u(x,y)$ be a nonconstant harmonic function in the disk $x^2+y^2 < R^2$. Define for each $0 < r < R$,

$$M(r) = \max_{x^2+y^2=r^2} u(x,y).$$

Prove that $M(r)$ is a monotone increasing function in the interval $(0,R)$.

3. (M, 20 points) Consider the heat equation:

$$u_t = ku_{xx}, \quad t > 0, \quad -\infty < x < \infty.$$

$$u(x,0) = \begin{cases} U_1 & x < 0, \\ U_2 & x > 0. \end{cases}$$

Here U_1 and U_2 are constants.

(a) Please solve the problem. Is the solution unique?

(b) Please find $\lim_{t\to\infty} u(x,t)$.

4. (M, 15 points) Use the energy method to prove uniqueness for the problem

$$u_{tt} - c^2u_{xx} + hu = F(x,t), \quad -\infty < x < \infty, t > 0.$$

$$\lim_{x\to\pm\infty} u(x,t) = \lim_{x\to\pm\infty} u_x(x,t) = \lim_{x\to\pm\infty} u_t(x,t) = 0, \quad t \geq 0.$$

$$\int_{-\infty}^{\infty} (u_t^2 + c^2u_x^2 + hu^2)dx < \infty, \quad t \geq 0,$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad -\infty < x < \infty,$$

where c and h are positive constants.

5. (E, 15 points) Please solve the initial boundary value problem

$$u_t + cu_x = -\lambda u, \quad x > 0, \ t > 0.$$

$$u(x,0) = 0, \quad x > 0; \quad u(0,t) = g(t), \ t > 0.$$

6. (M, 15 points) $f(x)$ and $g(x)$ are continuous functions satisfying $f(x+1) = f(x)$, $g(x+1) = g(x)$. Please prove that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = \left(\int_0^1 f(x)dx\right)\left(\int_0^1 g(x)dx\right).$$

7. (M, 15 points) Let $u(x) = u(x_1, \cdots, x_n) \in C^2$ for $|x| < a$. And $u \in C^0$ for $|x| \leq a$. u is a nonnegative solution of the Laplace equation:

$$\Delta u = 0, \quad |x| < a.$$

Please prove that

$$\frac{a^{n-2}(a-|\xi|)}{(a+|\xi|)^{n-1}}u(0) \leq u(\xi) \leq \frac{a^{n-2}(a+|\xi|)}{(a-|\xi|)^{n-1}}u(0).$$