PhD Qualify Exam in Numerical Analysis, March 21, 2018

1. (Average) Consider the initial value problem

$$\text{(I.V.P.)} \begin{cases} \frac{dy}{dt} = f(t, y) & \text{for } \; t \in (a, b), \\ y(a) = \alpha. \end{cases}$$

   Show that the difference method

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + a_1 f(t_i, w_i) + a_2 f(t_i + \delta, w_i + \beta f(t_i, w_i)), \end{aligned}$$

   for each $i = 0, 1, \ldots, N-1$, cannot have local truncation error $O(h^3)$ for any choice of a_1, a_2, δ and β. (10%)

2. (Easy) The iteration equation for the secant method can be written in the simpler form

$$x_n = \frac{f(x_{n-1})x_{n-2} - f(x_{n-2})x_{n-1}}{f(x_{n-1}) - f(x_{n-2})}.$$

   Explain why, in general, this iteration equation is likely to be less accurate than the one given by

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}.$$

   (10%)

3. (Average) Show that the numerical quadrature formula

$$Q(P) = \sum_{i=1}^{n} c_i P(x_i)$$

   can not accomplish an exact computation, provided that polynomial $P(x)$ is of degree greater than $2n - 1$, regardless of the choice of $c_1, c_2, \ldots, c_n$ and $x_1, x_2, \ldots, x_n$. (10%)

4. (Average) A sequence $\{p_n\}$ is said to be superlinearly convergent to p if

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0.$$

   (a) (7%) Show that if $p_n \to p$ of order α for $\alpha > 1$, then $\{p_n\}$ is superlinearly convergent to p.

   (b) (8%) Show that $p_n = \frac{1}{n^n}$ is superlinearly convergent to 0 but does not converge to 0 of order α for any $\alpha > 1$.

5. Let $x^{(k)} = Tx^{(k-1)} + c$, $k = 1, 2, \ldots$ with a given $x^{(0)}$ be an iterative method for solving the linear system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$.

   (a) (Average) Show that
   $$\|x^{(k)} - x\| \le \|T\|^k \|x^{(0)} - x\|$$
   and
   $$\|x^{(k)} - x\| \le \frac{\|T\|^k}{1 - \|T\|}\|x^{(1)} - x^{(0)}\|,$$
   where x is the fixed point of the iteration $x^{(k)} = Tx^{(k-1)} + c$, provided that $\|T\| < 1$. (10%)

   (b) (Easy) Show that the Jacobi iteration converges if A is strictly diagonally dominant. (5%)

   (c) (Average) Show that the Gauss-Seidel iteration converges to a solution of $Ax = b$ if A is strictly diagonally dominant or A is symmetric positive definite. (10%)

6. (Average) Suppose that $A \in \mathbb{R}^{n\times n}$ is nonsingular and that for specific given right hand side vectors $b, g \in \mathbb{R}^n$, solutions to linear systems $Ay = b$ and $Az = g$, respectively, are already known. Show how to solve the system
   $$\begin{bmatrix} A & g \\ h^T & \alpha \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} b \\ \beta \end{bmatrix}$$
   in $O(n)$ flops where $\alpha, \beta \in \mathbb{R}$ and $h \in \mathbb{R}^n$ are given and the matrix $A_+ = \begin{bmatrix} A & g \\ h^T & \alpha \end{bmatrix}$ is nonsingular. (10%)

7. (Average) Apply Householder reflection transformation to verify that
   $\det(I_n + xy^T) = 1 + x^Ty$, where $x, y \in \mathbb{R}^n$. (10%)

8. (Easy) Show that if B is singular, then
   $$\frac{1}{\kappa(A)} \le \frac{\|A - B\|}{\|A\|},$$
   where $\kappa(A) = \|A\|\|A^{-1}\|$. (10%)