Qualifying Exam in Partial Differential Equations

Spring 2018, NCKU Math

1.(E.10pt) Solve the linear Reaction-Diffusion-Advection equation

$$u_t = u + u_x + u_{xx} \quad , x \in \mathbb{R}, t > 0$$

with $u(x, 0) = g(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

2.(E.10pt) Solve the first order linear equation

$$y u_x + u_y = u \quad , (x, y) \in \mathbb{R}^2$$

with $u(x, 0) = x^2$.

3.(M.15pt) Given the Poisson formula in half-plane

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(z)}{(x - z)^2 + y^2} dz \quad , x \in \mathbb{R}, y > 0$$

with $g \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Given $x_0 \in \mathbb{R}$, show that $u(x_0, y) \to g(x_0)$ as $y \to 0^+$.

4.(M.15pt) Use the Fourier transform method to solve the heat equation

$$u_t = u_{xx} \quad , x \in \mathbb{R}, t > 0$$

with $u(x, 0) = g(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

5.(M.15pt) Let $u \in C^2$ be a solution of the damped wave (or telegraph) equation

$$u_{tt} + u_t = u_{xx} \quad , x \in \mathbb{R}, t > 0.$$

Given $x_0 \in \mathbb{R}$ and $t_0 > 0$, show that if $u(x, 0) = u_t(x, 0) = 0$ for all $x \in [x_0 - t_0, x_0 + t_0]$, then $u(x_0, t_0) = 0$.

6.(M.15pt) Find a traveling wave solution of the viscous Burgers' equation

$$u_t + u u_x = u_{xx} \quad , x \in \mathbb{R}, t > 0.$$

That is, $u(x, t) = v(x - ct)$ with $c \in \mathbb{R}$ and $v \in C^2(\mathbb{R})$ such that $v(s) \to 0$ as $s \to -\infty$ and $v(s) \to 1$ as $s \to +\infty$.

7.(D.20pt) Given the Newton potential in space

$$u(x) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{f(y)}{|x - y|} dy \quad , x \in \mathbb{R}^3$$

with $f \in C_c^2(\mathbb{R}^3)$. Show that $u \in C^2(\mathbb{R}^3)$ and $-\Delta u = f$ in $\mathbb{R}^3$.