PhD Qualify Exam

General Analysis

(E: Easy, M: Moderate, D: Difficult) October 18, 2017

1. (15 pts, M, 2008) Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let f be a positive integrable function on X. Prove that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any $A \in \mathfrak{M}$, if $\mu(A) \leq \delta$ then $\int_A f d\mu \leq \varepsilon$.

2. (10 pts, E, 2011) Let g be a nonnegative measurable function on $[0, 1]$ and $\int \log(g(t))dt$ is defined. Show that
$$exp\Big(\int_0^1 \log(g(t))dt \Big) \leq \int_0^1 g(t)dt.$$

3. (15 pts, E, 2013) Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $\{E_n\}$ be a sequence in $\mathfrak{M}$ with $E_{n+1} \subseteq E_n$ foa all n. If there exists some j such that $\mu(E_j) < \infty$. Show that $\mu(\cap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$. Give a counterexample if $\mu(E_j) = \infty$ for all j.

4. (15 pts, E, 2014) Let E be a measurable set in $\mathbb{R}^n$. f and f_k are measurable in E. If $p > 0$, and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, show that there is a subsequence $f_{k_j} \to f$ a.e. in E.

5. (15 pts, M) Let E be a measurable set in $\mathbb{R}^n$ and $1 \leq p < \infty$. Suppose $\{f_k\}$ is a sequence in $L^p(E)$ that converges pointwise a.e. on E to the function f which belongs to $L^p(E)$. Show that $\|f_k - f\|_{L^p(E)} \to 0$ as $k \to \infty$ if and only if $\|f_k\|_{L^p(E)} \to \|f\|_{L^p(E)}$ as $k \to \infty$.

6. (15 pts, E) Let E be a Lebesgue measurable set with finite measure. For $1 \leq p < \infty$, define
$$N_p[f] = \Big(\frac{1}{|E|} \int_E |f|^p \Big)^{1/p}.$$
Prove that if $p_1 < p_2$, then $N_{p_1}[f] \leq N_{p_2}[f]$.

7. (15 pts, E) Let $f \in L^p(E) \cap L^q(E)$ with $1 \leq p \leq q \leq \infty$ where E is a Lebesgue measurable set. Prove that $\|f\|_r \leq \|f\|_p^{\alpha} \|f\|_q^{1-\alpha}$ for all $p \leq r \leq q$, where $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.