PhD Qualify Exam: General Analysis

March, 2018

E: Easy; M: Moderate; D: Difficult.

Part A: ($15 \times 4 = 60$ points) Prove or disprove. Explain it.

(1) (E) Let $f_k, f : \mathbb{R} \to \mathbb{R}$ be real valued measurable functions and f_k converge to f in $L^p(\mathbb{R})$, where $1 < p < \infty$, then f_k converges to f in measure.

(2) (E) Let $|E| < \infty$ and $f \in L^p(E)$ for all $p > 1$, then

$$\lim_{p \to \infty} \|f\|_{L^p(E)} = \|f\|_{L^\infty(E)}\,.$$

(3) (E) Let f be a function of bounded variation, then f is an absolutely continuous function.

(4) (M) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function and improper Riemann integral on $[0, \infty)$, then f is Lebesgue integral on $[0, \infty)$.

Part B: ($20 \times 2 = 40$ points) Prove the following problems.

(5a) (E) Prove that for $a_j \geq 0, p_j > 1$, $j = 1, 2, \cdots N$ and

$$\sum_{j=1}^{N} \frac{1}{p_j} = 1\,,$$

we have

$$a_1 a_2 \cdots a_N \leq \sum_{j=1}^{N} \frac{a_j^{p_j}}{p_j}\,.$$

(5b) (M) Let $f : [0, 1] \to [0, \infty)$ be a continuous function, show that

$$\int_0^1 f^2(x)dx \leq \frac{1000}{3} + \frac{1}{15\sqrt{10}} \int_0^1 f^3(x)dx\,.$$

(6) (M) If $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise and there exists $M > 0$ such that $\|g_k\|_{L^\infty} < M$ for all k, prove that $f_k g_k \to fg$ in L^p.