Question:	1	2	3	4	5	6	7	Total
Points:	10	10	15	20	15	15	15	100
Score:								

Easier: 1,2 Medium: 3,4,5 Harder: 6,7

1. (10 points) Prove that any group of order 45 is not simple.
2. (10 points) Show that the ideal generated by 7 and $x^3 - 2$ in $\mathbb{Z}[x]$ is maximal.
3. (a) (10 points) Find all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
   (b) (5 points) Show that $\sqrt{5}$ is not an element $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
4. (a) (10 points) Prove that every PID is a UFD.
   (b) (10 points) Show that $\mathbb{Z}[\sqrt{-1}]$ is a UFD.
5. (15 points) Let F be a finite field of order p^n where p is a prime and n a positive integer. Prove that there is exactly one subfield of order p^m for each divisor m of n.
6. (15 points) Let M be a finite generated module over a commutative ring R with identity and $\phi : M \to M$ be an R-module homomorphism. Prove that there exists a polynomial $p(x)$ in $R[x]$ such that $p(\phi) = 0$ as an element in the R-algebra $\mathrm{Hom}_R(M, M)$ of R-module homomorphisms.
7. (15 points) Let p be a prime and let G be a finite group whose Sylow p-subgroup is normal. Show that the number of elements of order p in G is congruent to -1 modulo p.