PhD Qualify Exam in Numerical Analysis

March 16, 2017

1. (2011 Spring, Average) (10%) A forward-difference formula for $f'(x_0)$ can be expressed by

$$f'(x_0) = \frac{1}{h}\left[f(x_0+h) - f(x_0)\right] - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3)$$

Use extrapolation to derive an $O(h^3)$ formula for $f'(x_0)$.

2. (2015 Spring, Average) Let $A = \begin{bmatrix} 400 & 399 \\ 802 & 800 \end{bmatrix}$.

   (a) (10%) Compute A^{-1} and the condition number of A in the maximum norm, $\kappa_\infty(A)$.

   (b) (10%) Choose b, δb, x and δx such that

$$Ax = b, \qquad A(x+\delta x) = b + \delta b,$$

   and $\frac{\|\delta b\|_\infty}{\|b\|_\infty}$ is small, but $\frac{\|\delta x\|_\infty}{\|x\|_\infty}$ is large.

   (c) (5%) Choose b, δb, x and δx such that

$$Ax = b, \qquad A(x+\delta x) = b + \delta b,$$

   and $\frac{\|\delta x\|_\infty}{\|x\|_\infty}$ is small, but $\frac{\|\delta b\|_\infty}{\|b\|_\infty}$ is large.

3. (10%) Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular and that we have solutions to linear systems $Ax = b$ and $Ay = g$ where b, $g \in \mathbb{R}^n$ are given. Show how to solve the system

$$\begin{bmatrix} A & g \\ h^T & \alpha \end{bmatrix}\begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} b \\ \beta \end{bmatrix}$$

in $O(n)$ flops, where $\alpha, \beta \in \mathbb{R}$ and $h \in \mathbb{R}^n$ are given and the enlarged matrix $\begin{bmatrix} A & g \\ h^T & \alpha \end{bmatrix}$ is nonsingular.

and write the equations in the form $AU = F$. Show that $||A^{-1}||_2$ is uniformly bounded as $h \to 0$ and the numerical scheme is stable in the **2-norm**.

7. (2007 Spring, Easy) (15%) The following formulae are equivalent mathematically

$$\left(\sqrt{2}-1\right)^6 = \left(3-2\sqrt{2}\right)^3 = 99-70\sqrt{2}$$
$$= \frac{1}{\left(\sqrt{2}+1\right)^6} = \frac{1}{\left(3+2\sqrt{2}\right)^3} = \frac{1}{99+70\sqrt{2}}\,.$$

Please point out which one formula gives a minimal round-off error and explain why?