PhD Qualify Exam
General Analysis

March, 2017

(E:Easy, M:Moderate, D:Difficult)

1. (15 pts, E, 2016) Let E be a measurable set in $\mathbb{R}^n$. f and f_k are measurable in E and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, $0 < p \leq \infty$. Please prove that f_k converges to f in measure.

2. (10 pts, E) Let f be an absolutely continuous function on $[a, b] \subset \mathbb{R}$. If $Z \subset [a, b]$ is a Lebesgue measure-zero set, then $f(Z)$ has Lebesgue measure zero.

3. (20 pts, M, 2015) Please show the following *generalized dominated convergence theorem*: Let g be an integrable function, g_n be a sequence of integrable functions such that $g_n \to g$ a.e. and $|f_n| \leq g_n$, $f_n \to f$ a.e.. If
$$\int g dx = \lim_{n\to\infty} \int g_n dx,$$
then
$$\int f dx = \lim_{n\to\infty} \int f_n dx.$$

4. (20 pts, M, 2014) Let $\phi(x) \geq 0$ be a bounded measurable function in $\mathbb{R}^n$. $\phi = 0$ for $|x| \geq 1$ and $\int \phi dx = 1$. For $\epsilon > 0$, let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$.

   (a) If $f \in L^1(\mathbb{R}^n)$, show that $\lim_{\epsilon\to 0}(f * \phi_\epsilon)(x) = f(x)$ a.e..

   (b) If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, show that $\|(f * \phi_\epsilon)(x) - f(x)\|_p \to 0$ as $\epsilon \to 0$.

   (Recall: $(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy$.)

5. (20 pts, M) Prove the following *integral version of Minkowski's inequality* for $1 \leq p < \infty$:
$$\left[\int \left| \int f(x, y)dx \right|^p dy\right]^{1/p} \leq \int \left[\int |f(x, y)|^p dy\right]^{1/p} dx.$$

6. (15 pts, D) Let $(X, \mathcal{M}, \mu)$ be a positive measure space with $\mu(X) < \infty$, and let f and g be real-valued measurable functions.

   (i) (7 pts, E) Show that if
$$\int_E f d\mu = \int_E g d\mu, \quad \forall E \in \mathcal{M},$$
   then $f = g$ a.e..

(ii) (8 pts) If

$$\int_X f d\mu = \int_X g d\mu.$$

Show that either (a) $f = g$ a.e., or (b) there exists an $E \in \mathcal{M}$ such that

$$\int_E f d\mu > \int_E g d\mu.$$