Question:	1	2	3	4	5	6	7	Total
Points:	15	15	10	15	15	15	15	100
Score:								

Easier: 3,6 Medium: 1,2,5 Harder: 4,7

1. (15 points) Let $p \neq q$ be primes. Prove that any group of order p^2q is not simple.
2. (15 points) Prove that the symmetric group S_5 is not solvable.
3. Let $f : R \to S$ be a homomorphism between commutative rings with identity.
   (a) (5 points) Prove that $f^{-1}(P)$ is a prime ideal in R for any prime ideal P in S.
   (b) (5 points) Give an example to show that there exists a ring homomorphism $f : R \to S$ and a maximal ideal M of S such that $f^{-1}(M)$ is not a maximal ideal of R.
4. (15 points) Describe all the prime ideals of the polynomial ring $\mathbb{Z}[x]$ over the ring of integers.
5. (15 points) Let F be a finite field of order p^n where p is a prime and n a positive integer. Prove that there is exactly one subfield of order p^m for each divisor m of n.
6. (15 points) Let E be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Describe the subgroup lattice of the Galois group $\text{Gal}(E/\mathbb{Q})$ and the subfield lattice of E.
7. (15 points) Let M be a finite generated module over a commutative ring R with identity and $\phi : M \to M$ be an R-module homomorphism. Prove that there exists a polynomial $p(x)$ in $R[x]$ such that $p(\phi) = 0$ as an element in the R-algebra $\text{Hom}_R(M, M)$ of R-module homomorphisms.