PhD Qualify Exam
General Analysis

March 11, 2015

E: Easy, M: Moderate, D: Difficult

1. [E, 10%](Sept. 2011) Let f be Lebesgue measurable on $[0,1]$. Assume that
$$\int_0^1 [f(x)]^m dx = c, \text{ for all } m \in \mathbb{N}$$
where c is some constant. Show that $f = \chi_A$ a.e. for some $A \subset [0,1]$.

2. [M, 20%](Sept. 2012) **Determine whether the following statements are true or false. If true, prove it; if false, disprove it or give a counterexample.**

   Let f and f_k, $k = 1, 2, \cdots$ be measurable and finite a.e. in E, where $E \subset \mathbb{R}^n$ has a finite measure.
   (a) (10%) If f_k converges to f in measure, then f_k converges to f pointwise a.e.
   (b) (10%) If f_k converges to f pointwise a.e., then f_k converges to f in measure.

3. [M, 20%](Feb. 2006) Suppose $f \in L^1(\mathbb{R})$. Let $F(x) = \int_{\mathbb{R}} f(t)\frac{\sin xt}{t}dt$.
   (a) (10%) Prove that F is differentiable on $\mathbb{R}$ and find $F'(x)$.
   (b) (10%) Determine whether or not F is absolutely continuous on every compact subinterval of $\mathbb{R}$.

4. [E, 15%](Sept. 2004) Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Prove that $f * g \in L^p(\mathbb{R}^n)$, and
$$\|f * g\|_p \leq \|f\|_p \|g\|_1$$
where $(f * g)(x) = \int_{\mathbb{R}^n} f(t)g(x-t)dt$.

5. [M, 25%] Let $f \in L^1(\mathbb{R}^2)$ and Σ be the Lebesgue σ-algebra. For all $A \in \Sigma$, define $\mu_f(A) = \int_A |f| dx$.
   (a) (10%) Show that μ_f is a finite measure on $(\mathbb{R}^2, \Sigma)$.
   (b) (15%) Suppose a set $E \subset \mathbb{R}^2$ has outer measure $|E|_e < \infty$ and that for every $x \in E$, there exists a cube Q_x containing x such that
$$\mu_f(Q_x) \geq 0.25|Q_x|, \quad x \in E.$$
   Show that $|E|_e \leq 100\mu_f(\mathbb{R}^2)$.

6. [E, 10%] Fix $p \in (1, \infty)$. Find the maximal set $A \subset \mathbb{R}$ such that the following statement is true.
$$\alpha \in A \text{ and } f \in L^p([1,\infty)) \quad \Rightarrow \quad \left|\int_1^\infty \frac{f(x)}{x^\alpha}dx\right| < \infty.$$