國立成功大學應用數學所 數值分析 博士班資格考
October, 9, 2014

1. Let $u(x)$ be a function which interpolates $f(x)$ at $x_0, x_1, \ldots, x_{n-1}$, and $v(x)$ be a function which interpolates $f(x)$ at $x_1, x_2, \ldots, x_n$. Define $w(x)$ by
$$w(x) = \frac{(x_n - x)u(x) + (x - x_0)v(x)}{x_n - x_0}.$$
Show that the function $w(x)$ interpolates $f(x)$ at $x_0, x_1, \ldots, x_{n-1}, x_n$. (10%)

2. Let $x = (1, 4, 4, 6, 3, 0)^T$. Find a Householder transformation H and a positive number α so that $Hx = (0, 0, 4, 6, 0, \alpha)^T$. (10%)

3. Consider a linear system $Ax = b$ where
$$A = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix}$$
   (a) Choose the range of α so that A is positive definite. (5%)
   (b) Find a range of α so that Jacobi iteration converges. (10%)
   (c) Find a range of α so that Gauss-Seidel iteration converges. (10%)

4. Suppose the square linear system $Ax = b$ has been solved via adopting a partial pivoting factorization $PA = LU$ where P is a permutation matrix. If all computation is implemented in a t-digit decimal system and the condition number of A is about 10^q, for $1 < q < t$.
   (a) Estimate relative error of computed solution. (5%)
   (b) Can you give any strategy (an algorithm) to improve the accuracy of the numerical solution by using the same partial pivoting LU-factorization? (10%)
   (c) Can you give any strategy to improve the condition number of A so that the computed solution is more accurate? (10%)

5. Let K be the triangle with vertices $(0, 0)$, $(h, 0)$, and $(0, h)$. Find the element stiffness matrix A^K corresponding to the Poisson equation
$$\Delta u = f$$
using the linear functions $P^1(K)$. (15%)

6. Derive the variational formulation of the inhomogeneous Neumann problem

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma, \end{aligned}$$

where Γ is the boundary of Ω. Show that in the resulting variational formulation, the bilinear form is symmetric, V–elliptic and continuous, and the linear form is continuous if $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$. (15%)