Ph'D Qualifying Exam
General Analysis

March, 2014

E:Easy, M:Moderate, D:Difficult

1. (15 pts, M) Let $|\cdot|_e$ denote the outer measure. If $\{E_k\}$ is an increasing sequence of sets in R^n and $E_k \to E$, show that $\lim_{k\to\infty} |E_k|_e = |E|_e$.

2. (20 pts, M) Let $\phi(x) \geq 0$ be a bounded measurable function in R^n. $\phi(x) = 0$ for $|x| \geq 1$ and $\int \phi\, dx = 1$. For $\epsilon > 0$, let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$.

   (a) If $f \in L^1(R^n)$, show that $\lim_{\epsilon\to 0}(f \star \phi_\epsilon)(x) = f(x)$ a.e..

   (b) If $f \in L^p(R^n)$, $1 \leq p < \infty$, show that $\|(f \star \phi_\epsilon)(x) - f(x)\|_p \to 0$ as $\epsilon \to 0$.

   (Recall: $(f \star g)(x) = \int_{R^n} f(x-y)g(y)dy$.)

3. (15 pts, M) Let $f \in L^1(R^k)$. The maximal function $Mf(x)$ is defined as

$$Mf(x) = \sup_Q \frac{1}{|Q|}\int_Q |f(y)|dy.$$

   where the sup is taken over all cubes Q with center x.

   (a) Assume that both f and its maximal function Mf are in $L^1(R^k)$. Prove that $f(x) = 0$ a.e..

   (b) Define

$$f(x) = \begin{cases} x^{-1}(\log x)^{-2}, & 0 < x < 1/2. \\ 0, & \text{elsewhere.} \end{cases}$$

   Then $f \in L^1(R^1)$. Show that

$$Mf(x) \geq |2x\log(2x)|^{-1}, \text{ for } 0 < x < 1/4.$$

   so that $\int_0^1 Mf(x)dx = \infty$.

4. (15 pts, E, 2011) Let E be a measurable set in R^n. f and f_k are measurable in E. If $p > 0$, and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, show that there is a subsequence $f_{k_j} \to f$ a.e. in E.

5. (15 pts, E, 2013) Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < \infty$. For each $x \in E$, $|f_k(x)| \leq M_x$ for all k. Please show that given $\epsilon > 0$, there is a closed set $F \subset E$ and a finite M such that $|E - F| < \epsilon$ and $|f_k(x)| \leq M$ for all k and all $x \in F$.

6. (20 pts, M, 2012) Please prove or disprove

   (a) If f is strictly increasing continuous function with $f'(x) = 0$ a.e., then f is a constant function.

   (b) If f is an absolutely continuous function with $f'(x) = 0$ a.e., then f is a constant function.