PhD Qualifying exam on Mathematical Programming March 1,2013

1. (15 points) Recall that linear programming (LP) is a special case of the following conic optimization model

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b,\ x \in \mathcal{K}, \end{array}$$

where $\mathcal{K} \subseteq E^n$ is a prescribed closed convex cone. For example, $\mathcal{K} = \{x : x \geq 0\}$. Here we assume that A, b, c are proper dimensions and the rows in A are linearly independent. When $\mathcal{K} \triangleq \mathcal{K}_L$, the conic optimization model becomes the so-called " Second order cone Programming (SOCP)." When $\mathcal{K} \triangleq PSD(n)$, the conic optimization model becomes the so-called "Semidefinite Programming (SDP)". The popularity of SOCP is also due to that it is a generalized form of convex QCQP (Quadratically Constrained Quadratic Programming). To be precise, consider the following QCQP.

$$\begin{array}{ll} \text{minimize} & x^T Q_0 x + 2b_0^T x \\ \text{subject to} & x^T Q_i x + 2b_i^T x + c_i \leq 0,\ i = 1, 2, \cdots, m, \end{array}$$

where $Q_i \succeq 0$, i.e., Q is positive semidefinite for $i = 0, 1, 2, ..., m$.

   (a) (5 points) Given that $t \in E^1$ and $x \in E^n$, prove that

$$t \geq x^T x \text{ if and only if } \|(\frac{t-1}{2}, x)^T\| \leq \frac{t+1}{2}.$$

   (b) (10 points) Using the result of (a), please formulate QCQP as an SOCP problem.

2. (15 points) Let $f : S \rightarrow E_1$ be a concave function, where $S \subseteq E_n$ is a nonempty polytope with vertices $x_1, \cdots, x_E$. Show that the convex envelop of f over S is given by

$$f_S(x) = \min\{\Sigma_{i=1}^E \lambda_i f(x_i) : \ \Sigma_{i=1}^E \lambda_i x_i = x, \Sigma_{i=1}^E \lambda_i = 1, \lambda_i \geq 0, \text{ for } i = 1, 2, ..., E\}.$$

Hence, show that if S is a simplex in E_n, then f_S is an affine function that attains the same values as f over all the vertices of S.

3. (20 points) For the optimization problem of the form

$$\min\ f(x) \quad \text{s.t. } x \in \Omega,$$

where $f \in C^2$ and $\Omega \subseteq \mathbb{R}^n$. Let x^* be a relative minimum of f over Ω, and d be any feasible direction at x^*.

   (a) (8 points) State the first order necessary condition in terms of d and $\nabla f(x^*)$. Give an example to show that the condition is not sufficient.

   (b) (7 points) State and prove the second order necessary condition in terms of d; $\nabla f(x^*)$, and $\nabla^2 f(x^*)$.

   (c) (5 points) Consider the quadratic problem

$$\min\ \frac{1}{2} x^t Q x - f^t x \quad \text{s.t.} \quad Ax = b,$$

   where Q is a symmetric $n \times n$ matrix, $A \in \mathbb{R}^{m \times n}$, $f, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Prove or disprove that: if x^* is a local minimum point, then it must be a global minimum point.

4. (20 points) Let X be a nonempty open set in E_n, and consider $f : E_n \to E_1$, $g_i : E_n \to R$ for $i = 1, ..., m$, $h_i : E_n \to E_1$ for some $i = 1, ..., \ell$. Consider Problem P to

Minimize $f(x)$

subject to $g_i(x) \leq 0$ for $i = 1, 2, ..., m$,

$h_i(x) = 0$ for $i = 1, 2, ..., \ell$,

$x \in X$.

Let $\bar{x}$ be a feasible solution, and let $I = \{i : g_i(\bar{x}) = 0\}$. Suppose that the KKT conditions holds at $\bar{x}$, that is, there exist scalars $\bar{u}_i \geq 0$ for $i \in I$ and $\bar{v}_i$ for $i = 1, 2, .., , \ell$ such that

$$\nabla f(\bar{x}) + \sum_{i \in I} \bar{u}_i \nabla g_i(\bar{x}) + \sum_{i=1}^{\ell} \bar{v}_i \nabla h_i(\bar{x}) = 0.$$

(a) (10 points) Suppose that f is pseudoconvex at $\bar{x}$ and ϕ is quasiconvex at $\bar{x}$, where

$$\phi(x) = \sum_{i \in I} \bar{u}_i g_i(x) + \sum_{i=1}^{\ell} \bar{v}_i h_i(x).$$

Show that $\bar{x}$ is global optimal solution to Problem P.

(b) (10 points) Show that if $f + \sum_{i \in I} \bar{u}_i g_i + \sum_{i=1}^{\ell} \bar{v}_i h_i$ is pseudoconvex, then $\bar{x}$ is a global optimal solution to Problem P.

5. (20 points) Consider the bilinear program to minimize $c^t x + d^t y + x^t H y$ subject to $x \in X$ and $y \in Y$, where X and Y are bounded polyhedral sets in E_n and E_m, respectively. Let $\hat{x}$ and $\hat{y}$ be extreme points of the sets X and Y, respectively.

(a) (5 points) Verify that the objective function is neither quasiconvex nor quasiconcave.

(b) (5 points) Prove that there exists an extreme point $(\bar{x}, \bar{y})$ that solves the bilinear program.

(c) (5 points) Prove that the point $(\hat{x}, \hat{y})$ is a local minimum of the bilinear program if and only if the following are true:

(1) $c^t(x - \hat{x}) \geq 0$ and $d^t(y - \hat{y}) \geq 0$ for each $x \in X$ and $y \in Y$;

(2) $c^t(x - \hat{x}) + d^t(y - \hat{y}) > 0$ whenever $(x - \hat{x})^t H (y - \hat{y}) < 0$.

(d) (5 points) Show that the point $(\hat{x}, \hat{y})$ is a KKT point if and only if $(c^t + \hat{y}^t H)(x - \hat{x}) \geq 0$ for each $x \in X$ and $(d^t + \hat{x}^t H)(y - \hat{y}) \geq 0$ for each $y \in Y$.

6. (10 points) Let $f : S \to E_1$ be a continuous function, where S is a convex subset of E_n. Show that f is quasimonotone if and only if the level surface $\{x \in S : \ f(x) = \alpha\}$ is a convex set for all $\alpha \in E_1$.