GENERAL ANALYSIS

(E:Easy, M: Moderate, D: Difficult) **PhD Qualify Exam** **March 1, 2013**

1. (E, 15 pts, 2011, 6) Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $\{E_n\}$ be a sequence in $\mathfrak{M}$ with $E_{n+1} \subseteq E_n$ for all n. If there exists some j such that $\mu(E_j) < \infty$. Show that $\mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$. Give a counterexample if $\mu(E_j) = \infty$ for all j.

2. (E, 15 pts) Let $\{f_k\}$ be a sequence of measurable functions defined on a meaurable E with $|E| < \infty$. If $|f_k(x)| \le M_x < +\infty$ for all k for each $x \in E$, show that given $\varepsilon > 0$, there is a closed $F \subset E$ and a finite M such that $|E - F| < \varepsilon$ and $|f_k(x)| \le M$ for all k and all $x \in F$.

3. (E, 15 pts, 2011, 6) Let $f(x, y) = ye^{-xy} \sin x$ defined on a measurable set
$$E = \{(x, y) : 0 < x < \infty,\ 0 < y < 1\}.$$
Compute the integral
$$\iint_E ye^{-xy} \sin x dx dy$$
and justify your answer.

4. (M, 15 pts, 2011, 9) Let $(X, \mathfrak{M}, \mu)$ be a measure space. Assume that $f \in L^r(X)$ for some $0 < r < \infty$. Show that
$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

5. (M, 20 pts, 2012, 3) Let f be a bounded function on the closed, bounded interval $[a, b]$. Then f is Riemann integrable over $[a, b]$ if and only if the set of points in $[a, b]$ at which f fails to be continuous has measure zero.

6. (M, 20 pts) Let $g(x)$ be the function given by
$$g(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Determine if the point $x = 0$ is in the Lebesgue set of g or not. Aslo let $G(x) = \int_0^x g(s)\, ds$, $x \in \mathbb{R}^1$. Do we have $G'(0) = g(0)$ or not.