**PhD Qualifying exam on Mathematical Programming    September 28,2012**

1. (25 points) Prove or disprove the following statements.

   (a) (5 points) Let $P$ be a polyhedral set in $\mathbb{R}^n$. Assume that $P \neq \emptyset$ and $P \neq \mathbb{R}^n$. Then $\bar{x}$ is an extreme point of $P$ if and only if $P \setminus \{\bar{x}\}$ is a convex set.

   (b) (5 points) The optimum for maximizing a convex function over a bounded polyhedral set $P$ must be achieved at least on one of the extreme points of $P$.

   (c) (5 points) Consider the quadratic problem

$$\min \frac{1}{2}x^t Q x - f^t x$$
$$\text{s.t.} \quad Ax = b$$

   where $Q$ is symmetric $n \times n$ matrix, $A \in \mathbb{R}^{m \times n}$, $f$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. If $x^*$ is a local minimum point, then it must be a global minimum point.

   (d) (10 points) Given the following two linear programs:

$$\text{(P1)} \quad \min\ (c)^T x \quad \text{s.t. } Ax = b, x \geq 0,$$

$$\text{(P2)} \quad \min\ (c')^T x \quad \text{s.t. } Ax = b', x \geq 0,$$

   where $A \in \mathbb{R}^{m \times n}$, $c, c', x \in \mathbb{R}^n$, $b, b' \in \mathbb{R}^m$, $c' = \beta c$, $b' = \lambda b$, $\lambda > 0$ and $\beta \in \mathbb{R}$. Assume that (P1) has at least two feasible solutions but has a unique finite optimum. Moreover, (P1) is nondegenerate.

   i. (4 points) (P2) may be degenerate.

   ii. (3 points) (P2) may be unbounded.

   iii. (3 points) (P2) may have multiple optimal solutions.

2. (15 points) Recall that linear programming (LP) is a special case of the following conic optimization model

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b,\ x \in \mathcal{K}, \end{aligned}$$

   where $\mathcal{K} \subseteq E^n$ is a prescribed closed convex cone. For example, $\mathcal{K} = \{x : x \geq 0\}$. Here we assume that $A$, $b$, $c$ are proper dimensions and the rows in $A$ are linearly independent. When $\mathcal{K} \triangleq \mathcal{K}_L$, the conic optimization model becomes the so-called " Second order cone Programming (SOCP)." When $\mathcal{K} \triangleq PSD(n)$, the conic optimization model becomes the so-called "Semidefinite Programming (SDP)". The popularity of SOCP is also due to that it is a generalized form of convex QCQP (Quadratically Constrained Quadratic Programming). To be precise, consider the following QCQP.

$$\text{minimize} \qquad x^T Q_0 x + 2b_0^T x$$

subject to $\quad x^T Q_i x + 2b_i^T x + c_i \leq 0,\ i = 1, 2, \cdots, m,$

where $Q_i \succeq 0$, i.e., $Q$ is positive semidefinite for $i = 0, 1, 2, ..., m$.

(a) (5 points) Given that $t \in E^1$ and $x \in E^n$, prove that

$$t \geq x^T x \text{ if and only if } \|(\frac{t-1}{2}, x)^T\| \leq \frac{t+1}{2}.$$

(b) (10 points) Using the result of (a), please formulate QCQP as an SOCP problem.

3. (12 points) Let $f : S \to E_1$, where $S \subseteq E_n$ is a nonempty convex set. Then the convex envelop of $f$ over $S$, denoted $f_S(x)$, $x \in S$, is a convex function such that $f_S(x) \leq f(x)$ for all $x \in S$; and if $g$ is any other convex function for which $g(x) \leq f(x)$ for all $x \in S$, then $f_S(x) \geq g(x)$ for all $x \in S$. Hence, $f_S$ is the pointwise supremum over all convex underestimators of $f$ over $S$. Show that $\min\{f(x) : x \in S\} = \min\{f_S(x) : x \in S\}$, assuming that the minima exist, and that

$$\{x^* \in S : f(x^*) \leq f(x) \ \text{ for all } x \in S\} \subseteq \{x^* \in S : f_S(x^*) \leq f_S(x) \ \text{ for all } x \in S\}.$$

4. (13 points) Let $f : S \to E_1$ be a concave function, where $S \subseteq E_n$ is a nonempty polytope with vertices $x_1, \cdots, x_E$. Show that the convex envelop of $f$ over $S$ is given by

$$f_S(x) = \min\{\Sigma_{i=1}^E \lambda_i f(x_i) :\ \Sigma_{i=1}^E \lambda_i x_i = x, \Sigma_{i=1}^E \lambda_i = 1, \lambda_i \geq 0, \ \text{for } i = 1, 2, ..., E\}.$$

Hence, show that if $S$ is a simplex in $E_n$, then $f_S$ is an affine function that attains the same values as $f$ over all the vertices of $S$.

5. (15 points) Let $c$ be an $n$ vector, $b$ an $m$ vector, $\mathbf{A}$ an $m \times n$ matrix, and $\mathbf{H}$ a symmetric $n \times n$ positive definite matrix. Consider the following two problems:

- Minimize $\quad c^t x + \frac{1}{2} x^t \mathbf{H} x$
  subject to $\quad \mathbf{A} x \leq b$ ,
- Minimize $\quad h^t v + \frac{1}{2} v^t \mathbf{G} v$
  subject to $\quad v \geq 0,$

where $G = \mathbf{A}\mathbf{H}^{-1} A^t$ and $h = \mathbf{A}\mathbf{H}^{-1} c + b$. Investigate the relationship between the KKT conditions of these two problems.

6. (10 points) Let $S$ be aconvex set in $E^n$ and $S^*$ a convex set in $E^m$. Suppose $T$ ia an $m \times n$ matrix that establishes a one-to-one correspondence between $S$ and $S^*$, i.e., for every $s \in S$ there is $s^* \in S^*$ such that $Ts = s^*$, and for every $s^* \in S^*$ there is a single $s \in S$ such that$Ts = s^*$. Show that there is a one-to-one correspondence between extreme points of $S$ and $S^*$.

7. (10 points) Let

$$f(x) := \frac{1}{p} |x|^p,\ p > 1,\ x \in \mathbb{R}^n.$$

Compute the conjugate function $f^*$ and verify that $f^{**} = f$.