GENERAL ANALYSIS
PhD Qualify Exam. Sep. 28, 2012

(E: easy, M: moderate, D: difficult)

1. (E, 15 pts, 2004, 9) Let f and $f_k, k = 1, 2, \ldots$ be measurable and finite a. e. in E, where $E \subset \mathbb{R}^n$ has a finite measure. Prove that if $f_k \to f$ a. e. , then f_k converges to f in measure.

2. (E, 15 pts, 2007, 9) Show that every Lebesgue integrable function is the limit, almost everywhere, of a certain sequence of step functions.

3. (E, 15 pts, 2012, 3) Show that measurable sets are those which split every set (measurable or not) into pieces that are additive with respect to outer measure.

4. (M, 20 pts, 2007, 9) $1 < p < \infty, f \in L^p(0, \infty)$. Define

$$F(x) = \frac{1}{x}\int_0^x f(t)dt, \quad 0 < x < \infty$$

   (a) (15 pts) Prove that

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p$$

   (b) (5 pts) Prove that the equality holds only if $f = 0$ a.e..

5. (E, 20 pts)

   (a) (15 pts) Let E be a measurable set and $1 < p < \infty$. Suppose that $\{f_n\}$ converges weakly to f in $L^p(E)$. Then $\{f_n\}$ converges to f if and only if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$

   (b) (5 pts) Does the above statement hold for $p = 1$?

6. (E, 15 pts) A complete and totally bounded metric space is compact.