Algebra Qualifying Exam

March 2012

$\mathbb{Z}$ = integers. $\mathbb{Q}$ = rational numbers. $\mathbb{C}$ = complex numbers.
The level of difficulty of the problems is indicated by the number of $*$ with one $*$ indicates the easier problems.

1. $*$ Determine whether each statement below is true or false. If true, give a brief explanation. If false, provide a counterexample.
   (a) (5 points) All subgroups of a direct product $G \times H$ of groups G and H are of the form $G' \times H'$ for $G' \leq G$ and $H' \leq H$.
   (b) (5 points) An Artinian integral domain is a field.
   (c) (5 points) Every algebraic extension of a finite field is finite.
   (d) (5 points) If R is a PID, then its Jacobson radical $J(R)$ is zero. Recall that $J(R)$ is the intersection of all maximal ideals of R.
2. (10 points) $*$ Show that $\mathbb{Z}[\sqrt{-5}] = \{a + ib\sqrt{5} \mid a, b \in Z\}$ is not a unique factorization domain.
3. $*$ Let E be the splitting field over $\mathbb{Q}$ of the equation $x^4 - 5$.
   (a) (10 points) Determine the Galois group of E over $\mathbb{Q}$.
   (b) (5 points) Final all the intermediate fields K between E and $\mathbb{Q}$ satisfying $[E : K] = 2$.
4. $*$ Let X be a topological space. Consider the ring $R(X)$ of continuous real-valued functions on X. The ring structure is given by point-wise addition and multiplication.
   (a) (5 points) Show that for each $x \in X$ the set
   $$M_x = \{f \in R(X) \mid f(x) = 0\}$$
   is a maximal ideal in $R(X)$.
   (b) (5 points) Show that if X is compact, that is, every open covers of X has a finite subcover, then every maximal ideal in $R(X)$ is equal to M_x for some $x \in X$.
5. (15 points) $**$ Let V be a finite dimensional vector space over $\mathbb{C}$. Let $\phi : V \times V \to \mathbb{C}$ be a bilinear map satisfying $\phi(x, x) = 0$ for all $x \in V$. Assume that for any nonzero element $x \in V$, there is an element $y \in V$ such that $\phi(x, y) \neq 0$. Show that the dimension of V is even.
6. (15 points) $**$ Let $\mathbb{F}_q$ be the finite field with q elements and let $\mathrm{M}_2(\mathbb{F}_q)$ be the ring of 2×2 matrices over $\mathbb{F}_q$. Determine the number of nonzero nilpotent matrices in $\mathrm{M}_2(\mathbb{F}_q)$ as a function of q.
7. (15 points) $***$ Let $p \neq q$ be prime numbers. Prove that no group of order $p^2 q$ is simple.

This exam has 7 questions, for a total of 100 points.