國立成功大學應用數學所 數值分析 博士班資格考 **June, 17, 2011**

1. (99上) Consider the initial value problem

$$\text{(I.V.P.)} \begin{cases} y' = f(t, y), & a \le t \le b, \\ y(a) = \alpha. \end{cases}$$

   (a) (易) Show that

$$y'(t_i) = \frac{-3y(t_i) + 4y(t_{i+1}) - y(t_{i+2})}{2h} + \frac{h^2}{3} y'''(\xi_i),$$

   for some ξ_i with $t_i \le \xi_i \le t_{i+2}$. (5%)

   (b) (普) Part (a) suggests the difference method

$$w_{i+2} = 4w_{i+1} - 3w_i - 2hf(t_i, w_i), \quad \text{for} \quad i = 0, 1, \ldots, n-2.$$

   Analyze this method for consistency, stability and convergence. (15%)

2. (99下, 難) Consider the initial-value problem

$$y' = f(t, y), \quad \text{for} \quad a \le t \le b \quad \text{with} \quad y(a) = \alpha$$

   Let

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\phi(t_i, w_i, h) \quad \text{for} \quad i > 0, \end{aligned}$$

   and

$$\begin{aligned} \widetilde{w}_0 &= \alpha \\ \widetilde{w}_{i+1} &= \widetilde{w}_i + h\widetilde{\phi}(t_i, \widetilde{w}_i, h) \quad \text{for} \quad i > 0, \end{aligned}$$

   give two one-step methods for approximating solution of $y(t)$ with local truncation error $\tau_{i+1}(h) = O(h^n)$ and $\widetilde{\tau}_{i+1}(h) = O(h^{n+1})$, respectively, i.e.,

$$y(t_{i+1}) = y(t_i) + h\phi(t_i, y(t_i), h) + O(h^{n+1})$$

   and

$$y(t_{i+1}) = y(t_i) + h\widetilde{\phi}(t_i, y(t_i), h) + O(h^{n+2}).$$

   Please use these two methods to construct an adaptive step-size control method for the considering initial-value problem. (15%)

3. (91上, 易) Is it possible to use $af(x+h)+bf(x)+cf(x-h)$ with suitable chosen coefficients a, b, c to approximate $f''(x)$? How many function values are needed to approximate $f''(x)$? (10%)

4. (易) Suppose $S, T \in \mathbb{R}^{n\times n}$ are lower triangular matrices and that $(ST-\lambda I)x = b$ is a nonsingular system. Give an $O(n^2)$ algorithm for computing x. (10%)

5. (普) Let

$$A = \begin{bmatrix} \alpha & u^T \\ 0 & T \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

where $T \in \mathbb{R}^{2\times 2}$ contains a pair of complex conjugate eigenvalues of matrix A. Give an algorithm for computing an orthogonal matrix $Q \in \mathbb{R}^{3\times 3}$ such that

$$Q^T A Q = \begin{bmatrix} \widetilde{T} & v \\ 0 & \alpha \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

with $\lambda(\widetilde{T}) = \lambda(T)$. (15%)

6. (易) Let $x = (1, 0, 4, 6, 3, 4)^T$. Find a Householder transformation H and a positive number α so that $Hx = (1, \alpha, 4, 6, 0, 0)^T$. (10%)

7. (普) Explain how the single-shift QR step $H - \mu I = QR$, $\overline{H} = RQ + \mu I$ can be carried out implicitly. That is, show how the transition from $\overline{H}$ to H can be carried out without substracting the shift μ from the diagonal of H. (10%)

8. (普) Consider the abstract saddle point problem

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

Suppose that an Uzawa-type algorithm is adopted for solving the saddle point problem, that is, for x_0, y_0 given suitably, vectors $(x_i^T, y_i^T)^T$, $i = 1, 2, 3, \ldots$, are sequentially generated by

$$\begin{aligned} x_{i+1} &= x_i + Q_A^{-1}(f - (Ax_i + B^T y_i)) \\ y_{i+1} &= y_i + \tau(Bx_{i+1} - g) \end{aligned}$$

where Q_A is an approximation to A. Determine Q_A and τ so that the Uzawa-type algorithm is convergent. (10%)