國立成功大學應用數學所 數值分析 博士班資格考 March, 4, 2011

1. A forward-difference formula for $f'(x_0)$ can be expressed by

$$f'(x_0) = \frac{1}{h}\left[f(x_0+h) - f(x_0)\right] - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3)$$

   Use extrapolation to derive an $O(h^3)$ formula for $f'(x_0)$. (10%)

2. Suppose that $f(x)$ is a real-valued function having roots in a certain real interval. For finding a real root of $f(x)$, there is a well-known iterative method, named the secant method, which can be described by

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

   Show that the convergence of the secant method is of order $\alpha = \frac{1+\sqrt{5}}{2}$. (15%)

3. Let $A(\epsilon) \in \mathbb{R}^{n\times n}$ be a real matrix function of the real variable ϵ, that is, each element of matrix $A(\epsilon)$ is a function of ϵ. Show that the eigenvector $x(\epsilon)$ corresponding to a multiple eigenvalue $\lambda(\epsilon)$ can be not differentiable with respect to ϵ whenever all elements of matrix $A(\epsilon)$ and the eigenvalue $\lambda(\epsilon)$ are differentiable. (10%)

4. Show that the quadrature formula $Q(P) = \sum_{i=1}^{n} c_i P(x_i)$ cannot have degree of precision greater than $2n-1$, regardless of the choice of $c_1, \ldots, c_n$, and $x_1, \ldots, x_n$. (15%)

5. Consider the square linear system $Ax = b$. Let D be diagonal consisting of the diagonals of A. The parametric Jacobi method, called the relaxation of Jacobi iteration (JOR), is expressed by

$$x_{k+1} = x_k - \omega D^{-1}(Ax_k - b).$$

   Show that if Jacobi iteration converges then JOR converges for $0 < \omega \le 1$. (10%)

6. Suppose the square linear system $Ax = b$ has been solved via the partial pivoting factorization $PA = LU$. Assume t-digit, base $\beta = 10$ floating point arithmetic is used and the condition number of A, $\kappa_\infty(A)$, is about 10^q.

   a.) Estimate relative error of computed solution. (5%)

   b.) Can you give any strategy (a numerical method) to improve the accuracy of the numerical solution by using the same partial pivoting LU-factorization? (10%)

   c.) Can you give any strategy to improve the condition number of A so that the computed solution is more accurate? (10%)

7. Consider the initial-value problem

$$y' = f(t, y), \quad \text{for } a \le t \le b \text{ with } y(a) = \alpha$$

Let

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\phi(t_i, w_i, h) \quad \text{for } i > 0, \end{aligned}$$

and

$$\begin{aligned} \tilde{w}_0 &= \alpha \\ \tilde{w}_{i+1} &= \tilde{w}_i + h\tilde{\phi}(t_i, \tilde{w}_i, h) \quad \text{for } i > 0, \end{aligned}$$

give two one-step methods for approximating solution of $y(t)$ with local truncation error $\tau_{i+1}(h) = O(h^n)$ and $\tilde{\tau}_{i+1}(h) = O(h^{n+1})$, respectively, i.e.,

$$y(t_{i+1}) = y(t_i) + h\phi(t_i, y(t_i), h) + O(h^{n+1})$$

and

$$y(t_{i+1}) = y(t_i) + h\tilde{\phi}(t_i, y(t_i), h) + O(h^{n+2}).$$

Please use these two methods to construct an adaptive step-size control method for the considering initial-value problem. (15%)