General Analysis PhD Qualify Exam

June 17, 2011

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(E: easy, M:moderate, D:difficult) The following abstract measure space is denoted by $(\mathbf{X}, \mathfrak{M}, \mu)$ and $(\mathbf{R}, \mathfrak{L}, m)$ is the Lebesque measure space.

1. (E,15%) Let $\{E_n\}_1^\infty$ be a sequence in $\mathfrak{M}$ with $E_{n+1} \subseteq E_n$ for all n. If there exists j such that $\mu(E_j) < \infty$. Show that $\mu(\bigcap_{n=1}^\infty E_n) = \lim_{n\to\infty} \mu(E_n)$. Give a counterexample if $\mu(E_j) = \infty$ for all j.

2. (E,15%, Feb.2008)

   (a) Given an example of a function that is in $L^2(\mathbf{R})$ but not in $L^1(\mathbf{R})$.

   (b) Given an example of a function that is in $L^1((0,1))$ but not in $L^2((0,1))$.

   (c) Prove that any function $f \in L^1(I) \cap L^2(I)$ for any interval $I \subseteq \mathbf{R}$ must be in $L^p(I)$ for all p between 1 and 2.

3. (E,10%, Sept.2004) Let f and f_k, $k = 1, 2, \ldots$, be measurable and finite a.e. on E. Prove that if $f_k \to f$ a.e. on E and $|E| < \infty$, then $f_k \to f$ in measure on E.

4. (M,15%, Sept.2005)

   (a) Let g be a nonnegative measurable function on $[0,1]$ and $\int \log(g(t))\,dt$ is defined. Show that
   $$exp(\int \log(g(t))\,dt) \leq \int g(t)\,dt.$$

   (b) Explain the inequality as the arithmetic mean greater than the geometric mean. i.e. for $\xi_1, \ldots, \xi_n > 0$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, we have
   $$\Pi_{i=1}^n \xi_i^{\lambda_i} \leq \sum_{i=1}^n \lambda_i \xi_i.$$

5. (M,15%) State *Fatou's Lemma* and use it to get the *generalization of Dominated Convergence Theorem* : Let $\{g_n\}$ be a sequence of integrable functions which converge a.e. to an integrable g. Let $\{f_n\}$ a sequence of measurable functions such that $|f_n| \leq g_n$ and $\{f_n\}$ converges to f a.e. If $\lim \int g_n = \int g$, then $\lim \int f_n = \int f$.

6. (M,15%) Evaluate the double integral
   $$\int_{(0,\infty)\times(0,1)} y \sin x e^{-xy}\,dx\,dy.$$

7. (M,15%) Suppose M is a closed subspace of a *Hilbert* space H and $x_0 \in H$. Prove that
   $$\min\{||x - x_0|| : x \in M\} = \max\{|\langle x_0, y\rangle| : y \in M^\perp, \|y\| = 1\}.$$