General Analysis PhD Qualify Exam. March 4, 2011

(E: easy, M:moderate, D:difficult)

The following general measure space is denoted by $(\mathbf{X}, \mathfrak{M}, \mu)$.

1. (E, 10%) The *Cantor* set $C = [0,1]\setminus[(\frac{1}{3}, \frac{2}{3})\cup(\frac{1}{9}, \frac{2}{9})\cup(\frac{7}{9}, \frac{8}{9})\cup\cdots]$. Explain that the *Cantor* set C is a measure zero *Borel* set and the cardinal number $|C| = |\mathbf{R}|$ is uncountable.

2. (E, 15%) State *Hölder's* and *Minkowski's* inequalities. Prove one of them.

3. (M, 15%) (a) Let $\mathbf{X} \neq \emptyset$, $\mathcal{E}$ be a collection of subsets of $\mathbf{X}$ and $\mathfrak{M}(\mathcal{E})$ be the σ-algebra generated by $\mathcal{E}$. Give a sufficient condition on $\mathcal{E}$ such that an additive set function on it can uniquely extend a measure on $\mathfrak{M}(\mathcal{E})$.
(b) For example, let $\mathbf{X} = \{a, b, c, d\}$ be a four-element set, let $\mathcal{E} = \{\{a, b\}, \{b, c\}\}$, then $\mathfrak{M}(\mathcal{E}) =$? Give two measures $\mu \neq \nu$ on $\mathfrak{M}(\mathcal{E})$ but $\mu(E) = \nu(E)$, for all $E \in \mathcal{E}$ and $\mu(\mathbf{X}) = \nu(\mathbf{X}) < \infty$.

4. (M, 20%, 2008,2) Suppose that $\{f_n\}_{n\in\mathbf{N}}$ is a sequence of functions in $L^1(\mu)$, converging almost everywhere to an $L^1(\mu)$ function f. Suppose also that $\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1$.
(a) Prove that for every measurable set A, $\lim_{n\to\infty} \int_A |f_n|\, d\mu = \int_A |f|\, d\mu$.
(b) Prove that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

5. (M, 10%, 2004,9) If $p > 0$ and $\int_{\mathbf{X}} |f - f_n|^p\, d\mu \to 0$ as $n \to \infty$, show that $\{f_n\}$ converges in measure on $\mathbf{X}$ to f.

6. (M, 15%, 2006, 2) Let $(\mathbf{X}, \mathfrak{M}, \mu)$ be a σ-finite measure space. Suppose that ν be a finite measure defined on $\mathfrak{M}$ such that ν is absolutely continuous with respect to μ. Let g be the *Radon-Nikodym* derivative of ν with respect to μ. Prove that $\int_{\mathbf{X}} f\, d\nu = \int_{\mathbf{X}} fg\, d\mu$.

7. (M, 15%) Let H be a *Hilbert* space, and let M be a non-trivial closed subspace of H, i.e. $M \neq \{0\}$. Let P_M be the orthogonal projection of H onto M, i.e. for any $x \in H$, we have $P_M x \in M$ and $x - P_M x \in M^{\perp}$. You may assume without proof that P_M is linear. Show that the operator norm $\|P_M\|_{op} = 1$, $P_M^2 = P_M$ and $P_M = P_M^*$. (i.e. orthogonal projections are linear idempotent self-adjoint contractions)