General Algebra PhD Qualifying Exam Date: Fri 4/3/2011

You may quote any standard results without proving them, but state clearly what facts you are assuming. Answers without explanation may receive no credit. Do all the problems. In the following, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ is the field of real numbers, and $\mathbb{C}$ is the field of complex numbers.

1. Let G be a group of order $10,989$ (note that $10989 = 3^3 \cdot 11 \cdot 37$).
   (a) (6%) Compute the number, n_p, of Sylow p-subgroups for each of $p = 3$, 11, and 37; for each of these n_p give the order of the normalizer of a Sylow p-subgroup.
   (b) (7%) Prove that G contains either a normal Sylow 37-subgroup or a normal Sylow 3-subgroup.
   (c) (5%) Explain why (in all cases) G has a normal Sylow 11-subgroup.
2. (a) (6%) Let N be a normal subgroup of a group G. Prove that if both N and G/N are solvable, then G is solvable.
   (b) (6%) Show that the symmetric group S_n is not solvable for $n \geq 5$.
3. (10%) Prove that $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain.
4. Let R be a commutative integral domain with 1. A nonzero, nonunit element $s \in R$ is said to be "special" if, for every element $a \in R$, there exist q, $r \in R$ with $a = qs + r$ and such that r is either 0 or a unit of R.
   (a) (6%) If $s \in R$ is special, prove that the principal ideal (s) generated by s is maximal in R.
   (b) (4%) Show that every polynomial in $\mathbb{Q}[X]$ of degree 1 is special in $\mathbb{Q}[X]$.
   (c) (10%) Prove that there are no special elements in the polynomial ring $\mathbb{Z}[X]$. (Hint: Apply the definition of special with $a = 2$ and with $a = X$.)
5. Let $n \geq 2$ be a positive integer, and let Ψ_n denote the set of all primitive n-th roots of unity in $\mathbb{C}$, i.e., Ψ_n is the set of generators of the group of n-th roots of unity in $\mathbb{C}$. Define
$$\Phi_n(x) := \prod_{\zeta \in \Psi_n} (x - \zeta).$$
   (a) (5%) Show that
$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$
   where the product is over all divisors d of n.
   (b) (5%) Find $\Phi_{12}(x)$.
   (c) (5%) Prove that $\Phi_n(x)$ is a monic polynomial in $\mathbb{Z}[x]$ with the degree equal to the Euler phi-function $\varphi(n)$.
   (d) (5%) Prove that for n odd, $n > 1$, $\Phi_{2n}(x) = \Phi_n(-x)$

(OVER)

6. Let E be a splitting field of the polynomial $x^3 - 2$ over the rationals $\mathbb{Q}$, and assume that E is contained in the complex number $\mathbb{C}$. Let $F = E \cap \mathbb{R}$ be the real subfield of E, and note that $F = \mathbb{Q}[\sqrt[3]{2}]$.

   (a) (6%) Show that $G = \mathrm{Gal}(E/\mathbb{Q})$ contains an element σ with the property that the only elements of F fixed by σ are rational.

   (b) (8%) Let $a \in F$ and suppose that $a^3 \in \mathbb{Q}$. Show that one of a, $a\sqrt[3]{2}$, or $a\sqrt[3]{4}$ is contained in $\mathbb{Q}$.

   (c) (6%) Prove that $\sqrt[3]{3} \notin E$.