PhD Qualify Exam, PDE, Mar. 04, 2011

Show all works

E: easy, M: moderate, D: difficult

1.(M) Find the solution for the problem [15%]

$$\begin{cases} u_t - u_{xx} = 0, & \\ u(x,0) = 0, & x > 0,\ t > 0, \\ u(0,t) = h(t), & \end{cases} \tag{1}$$

where $h(0) = 0$.

2.(M) Use the Fourier transform method to solve the initial value problem $\begin{cases} u_t = u_{xx}, & x \in R,\ t > 0, \\ u(x,0) = f(x), & x \in R. \end{cases}$
And prove that u satisfies the following inequality, for $1 \le q \le p \le \infty$ and $t > 0$, [15%]

$$\|u(\cdot,t)\|_{L^p(R)} \le (4\pi t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^q(R)}.$$

3.(M) Let $\lambda \in R$, $a > 0$, and u be a smooth function defined on a neighborhood of $D = \{(x,y) \in R^2 \,|\, x^2 + y^2 \le 1\}$ such that $\begin{cases} \Delta u + \lambda u = 0, & \text{in } x^2 + y^2 < 1, \\ \partial u/\partial n = -au, & \text{on } x^2 + y^2 = 1, \end{cases}$ where n is the unit outward normal vector to ∂D. Prove that if u is not identically zero in $x^2 + y^2 < 1$, then $\lambda > 0$. [15%]

4.(E) Let $D = \{\mathbf{x} \in R^3 \,|\, z > 0\}$ and $\mathbf{x}_0 = (x_0, y_0, z_0) \in D$. Let $u(\mathbf{x}_0) = \dfrac{z_0}{2\pi}\displaystyle\iint_{\partial D} \dfrac{h(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_0|^3}\, dS$. Show that $u(x_0, y_0, z_0) \to h(x_0, y_0)$ as $z_0 \to 0$. [15%]

5.(E) Let $F : R^n \times R \times R^n$ be a smooth function of (p, z, x). Assume that u is a smooth solution to $F(Du, u, x) = 0$ and $x(s)$ is a smooth curve. Prove that if $p(s) = Du(x(s))$, $z(s) = u(x(s))$, and $x_i'(s) = \dfrac{\partial F}{\partial p_i}$, then $p_i'(s) = \dfrac{\partial F}{\partial x_i} - \dfrac{\partial F}{\partial z} p_i$. [15%]

6.(E) By using the method of characteristics, find an explicit local solution to $u_t + \dfrac{1}{2}\Big((u_x)^2 + x^2\Big) = 0$ if $t > 0$, $x \in R$, with initial condition $u(x,0) = x^2/2$. [15%]

7.(D) Solve the wave equation for infinite vibrating string $u_{tt} = c^2(x) u_{xx}$, where $c(x) = \begin{cases} c_1, & x < 0. \\ c_2, & x > 0. \end{cases}$
Let a wave $u(x,t) = f(x - c_1 t)$ come in from the left, see the figure below. Thus the initial conditions are $u(x,0) = f(x)$ and $u_t(x,0) = -c_1 f'(x)$. Assume that $u(x,t)$ and $u_x(x,t)$ are continuous everywhere. Also give an interpretation for the solution you find. [10%]

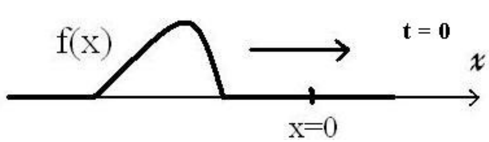

Figure 1: