General Analysis, Ph.D. Qualifying Exam, September 23, 2011

(E: easy, M:moderate, D:difficult)

1. (E, 10 pts) Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz transformation, i.e., there exists a constant c such that
$$\|Tx - Ty\| \le c\,\|x-y\| \quad \text{for all } x,\ y \in \mathbb{R}^n.$$
Let $E \subset \mathbb{R}^n$ be a measurable set. Is $T(E)$ measurable? Prove or disprove.

2. (M, 15 pts) Let f be upper semicontinuous and less than $+\infty$ on a compact set $E \subset \mathbb{R}^n$. Show that f is bounded above, and f also assumes its maximum on E.

3. (M, 15 pts) Let f be Lebesgue measurable on $[0,1]$. Assume that
$$\int_0^1 [f(x)]^m\,dx = c, \text{ for all } m \in \mathbb{N},$$
where c is some constant. Show that $f = \chi_A$ a.e. for some $A \subset [0,1]$.

4. (M, 20 pts) Let $(X, \mathfrak{M}, \mu)$ be a measure space. Assume $f \in L^r(X)$ for some $r < \infty$. Then
$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty .$$

5. (D, 20 pts, 2010, 3) $1 < p < \infty$, $f \in L^p(0,\infty)$. Define
$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad 0 < x < \infty.$$
(a) Prove that
$$\|F\|_p \le \frac{p}{p-1}\|f\|_p .$$
(b) Prove that the equality holds only if $f = 0$ a.e.

6. (E, 10 pts, 2011, 6) Let g be a nonnegative measurable function on $[0,1]$ and $\int \log(g(t))\,dt$ is defined. Show that
$$\exp\left(\int \log(g(t))\,dt\right) \le \int g(t)\,dt.$$

7. (E, 10 pts, 2005, 9) Compute the limit
$$\lim_{n\to\infty}\int_0^\infty \frac{n\cos x}{1+n^2x^{3/2}}dx.$$