國立成功大學應用數學所 數值分析 博士班資格考
September, 23, 2011

1. (91上,Easy) (10%); Is it possible to use $af(x+h)+bf(x)+cf(x-h)$ with suitable chosen coefficients a, b, c to approximate $f''(x)$? How many function values are needed at least to approximate $f''(x)$?

2. (94下,Average) (10%); Find the constants A_0, A_1, A_2, x_0, x_1 and x_2 such that the Gaussian quadrature rule

$$\int_{-1}^{1} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2)$$

   is exact for $f(x)$ in Π_5 which stands for the set of polynomials of degree less than or equal to 5.

3. (97上,Easy) (10%); Write an efficient algorithm for evaluating

$$u = \sum_{i=1}^{n} \prod_{j=1}^{i} d_j$$

4. (99上,Average); Consider a linear system $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix}$$

   (a) Choose the range of α so that A is positive definite. (5%)

   (b) Find a range of α so that Jacobi iteration converges. (5%)

   (c) Find a range of α so that Gauss-Seidel iteration converges. (5%)

5. (Easy);

   (a) If a numerical solution converges to the exact solution in the max-norm, does it converge in the 1-norm? (5%)

   (b) If a numerical solution converges to the exact solution in the 1-norm, does it converge in the max-norm? (5%)

6. (Difficult) (10%); Consider the linear boundary value problem with Dirichlet boundary conditions:

$$u''(x) = f(x) \quad \text{for } 0 < x < 1$$
$$u(0) = \alpha, \quad u(1) = \beta.$$

We attempt to compute a grid function consisting of values U_0, U_1, ..., U_m, U_{m+1} where U_j is our approximation to the solution $u(x_j)$. Here $x_j = jh$ and $h = 1/(m+1)$ is the mesh width, the distance between grid points. From the boundary conditions we know that $U_0 = \alpha$ and $U_{m+1} = \beta$ and so we have m unknown values U_1, ..., U_m to compute. We solve this problem with a centered difference scheme,

$$\frac{1}{h^2}(U_{j-1} - 2U_j + U_{j+1}) = f(x_j) \quad \text{for } j = 1, 2, \ldots, m.$$

The problem can be written in the form $AU = F$ where U is the vector of unknowns $U = [U_1, U_2, \ldots, U_m]^T$ and

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}, \qquad F = \begin{bmatrix} f(x_1) - \alpha/h^2 \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{bmatrix}.$$

Show that $||A^{-1}||_\infty$ is uniformly bounded as $h \to 0$ and the numerical scheme is stable in the **max-norm**.

7. (Easy) (10%); Consider the linear boundary value problem with Dirichlet boundary conditions:

$$u''(x) = f(x) \quad \text{for } 0 < x < 1$$
$$u(0) = \alpha, \quad u(1) = \beta.$$

We first solve the problem using the second order difference method and the numerical scheme can be written as $AU = F$ where U is the vector of unknowns $U = [U_1, U_2, \ldots, U_m]^T$ and U_j is our approximation to the solution $u(x_j)$.

The local truncation error of the numerical scheme is

$$\tau_j = \frac{1}{h^2}(u(x_{j-1}) - 2u(x_j) + u(x_{j+1})) - f(x_j) = \frac{1}{12}h^2 u''''(x_j) + O(h^4).$$

(a) Suppose that $f(x)$ is sufficiently smooth and given explicitly, use the method of deffered corrections to derive a fourth order scheme.

(b) Suppose that we only have the value of $f(x)$ at the grid points (but we know that the underlying function is sufficiently smooth), use the method of deffered corrections to derive a fourth order scheme.

8. (Average) (10%); Consider the Poisson problem with Dirichlet boundary conditions:

$$\begin{aligned} &u_{xx} + u_{yy} = f(x,y) \quad \text{for } 0 < x, y < 1, \\ &u(0,y) = \alpha_0(y), \quad u(1,y) = \alpha_1(y), \\ &u(x,0) = \beta_0(x), \quad u(x,1) = \beta_1(x), \end{aligned}$$

We attempt to compute a grid function consisting of values $U_{0,0}$, $U_{1,0}$, ..., $U_{m+1,m}$, $U_{m+1,m+1}$ where $U_{i,j}$ is our approximation to the solution $u(x_i, y_j)$. Here $x_i = ih$, $y_j = jh$ and $h = 1/(m+1) = \Delta x = \Delta y$. Solve this problem with a centered difference scheme,

$$\frac{1}{h^2}(U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j}) = f_{i,j} = f(x_i, y_j) \quad \text{for } i, j = 1, 2, \dots, m,$$

and write the equations in the form $AU = F$. Show that $||A^{-1}||_2$ is uniformly bounded as $h \to 0$ and the numerical scheme is stable in the **2-norm**.

9. (Easy) (15%); Consider the initial value problem:

$$\frac{du}{dt} = f(u(t)), \quad u(0) = u_0,$$

where$t \in [0,T]$ is the time variable, $T > 0$, $u \in \mathbb{R}$ is a real-valued function, and the function $f \in \mathbb{R}$ is assumed to be Lipschitz continuous with respect to u for $t \in [0,T]$, yielding the existence and uniqueness of the solution for this problem.

Denote U^n to be the numerical approximation of u at time, $t_n = nk$, where k is the time step. Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

(a) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$

(b) $U^{n+1} = U^n$

(c) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$

(d) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.