MIDTERM 1 FOR NUMBER THEORY

Date: Thursday, Nov 23, 2000
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No credit will be given for an answer without reasoning.

1.

(i) [5%] Prove that $n^2 - 1$ is divisible by 8 if n is odd.
(i) [5%] Prove that $n^3 - n$ is divisible by 6 for every integer n.

2. [10%] Suppose that m is odd. Show that

$$\frac{-(m-1)}{2}, \frac{-(m-3)}{2}, \ldots\ldots, \frac{m-3}{2}, \frac{m-1}{2}$$

is a complete residue system modulo m.

3. [10%] Solve the congruence equation $x^2 + x + 7 \equiv 0 \pmod{27}$.

4. [10%] Let p be a prime and let $\gcd(a,p) = \gcd(b,p) = 1$. Suppose that $x^2 \equiv a \pmod{p}$ and $x^2 \equiv b \pmod{p}$ are not solvable. Show that $x^2 \equiv ab \pmod{p}$ is solvable.

5.

(i) [5%] Evaluate the Legendre symbol $\left(\frac{-35}{101}\right)$.
(ii) [5%] Find the number of integers in the set $S = \{1, 2, \ldots, 2100\}$ that are divisible by neither 3 nor 5.

6. [10%] If n is an even integer, show that $\sum_{d|n} \mu(d)\phi(d) = 0$ where $\mu(d)$ is the Möbius function and $\phi(d)$ is the number of elements in a reduced residue system modulo d.

7. [10%] Suppose that f and g are multiplicative arithmetic functions. Show that $f * g$ is also multiplicative.

8. [10%] Prove that if $x^2 + y^2 = z^2$, then one of x, y is a multiple of 3 and one of x, y, z is a multiple of 5.

9. [10%] Two quadratic forms $f(x, y)$ and $g(X, Y)$ are said to be equivalent if there is an integral transformation with determinant ± 1 that carries $f(x, y)$ to $g(X, Y)$. Prove that the following quadratic forms

$$ax^2 + bxy + cy^2, \qquad ax^2 - bxy + cy^2, \qquad cx^2 + bxy + ay^2$$

are equivalent.

10. [10%] Let p be a prime and k be an integer. Prove that

$$1^k + 2^k + \cdots + (p-1)^k \equiv \begin{cases} 0 \pmod{p}, & \text{if } k \text{ is not divisible by } p-1; \\ -1 \pmod{p}, & if\ k \text{ is divisible by } p-1. \end{cases}$$