編號： 46 系所：數學系應用數學 科目：線性代數

本試題是否可以使用計算機： □可使用 ， ☑不可使用 （請命題老師勾選）

1. Let $M_2(\mathbb{R})$ be the set of all 2×2 real matrices and let

$$L = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \middle| \, a + d = 0 \right\}.$$

(a) (5 %) Show that L is a subspace of $M_2(\mathbb{R})$.
(b) (5 %) Find the dimension of L.

2. Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that the matrix of S with respect to the standard basis is given by

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) (5 %) Show that S is an isomorphism.

(b) (5 %) Find the matrix of S with respect to the basis $\{(1,1,0), (0,1,-1), (1,1,1)\}$.

3. (10 %) Let V be a vector space over $\mathbb{R}$. Let $\alpha : V \to V$ be a linear transformation such that $\alpha^3 = \alpha$. Show that $V = W_0 \oplus W_1 \oplus W_{-1}$, where $W_0 = \ker \alpha$, $W_1 = \{v \in V |\, \alpha(v) = v\}$ and $W_{-1} = \{v \in V |\, \alpha(v) = -v\}$.

4. Let $\{v_1, v_2, v_3, v_4\}$ be a basis of a vector space V over a field F. Determine if the following set is a basis of V. Justify your answer.

(a) (5%) $\{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\}$.

(b) (5%) $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1\}$.

5. (10 %) Let A be an $n \times n$ matrix over $\mathbb{R}$. Suppose that $\operatorname{tr} A^k = 0$ for all positive integers k. Is $A = 0$? Justify your answer.

6 . Let X_1 and X_2 be subspaces of a finite dimensional vector space V of dimension n.

(a) (7 %) Suppose that both X_1 and X_2 are both of dimension $n-1$ and $X_1 \neq X_2$. What is the dimension of $X_1 \cap X_2$? Justify your answer.

(b) (3 %) Let $X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\, x_1 + x_2 + x_3 + x_4 = 0\}$ and $Y = \{(x, y, x, y) \in \mathbb{R}^4 |\, x, y \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^4$. What is the dimension of $X \cap Y$?

（背面仍有題目,請繼續作答）

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7. (10 %) Let $V^* = \{f : V \to F | f \text{is linear }\}$ be the dual space of V. For any linear transformation $T : V \to V$, we define a linear transformation $T^* : V^* \to V^*$ by

$$T^*(f) = f \circ T \qquad \text{for any } \; f \in V^*.$$

Suppose that the matrix of T with respect to a basis $B = \{x_1, x_2, \ldots, x_n\}$ is given by $A = (a_{ij})_{1 \le i,j \le n}$. Show that the matrix of T^* with respect to the dual basis $B^* = \{x_1^*, x_2^*, \ldots, x_n^*\}$ is the transpose of A.

8. (15 %) Let V be an n-dimensional vector space over a field F and let $f : V \to V$ be a linear transformation. Suppose that the minimal polynomial of f is given by $(x-\lambda)^n$ for some $\lambda \in F$. Show that there is a basis $\{v_1, \ldots, v_n\}$ of V such that

$$f(v_1) = \lambda v_1 \quad \text{and} \quad f(v_i) \in \text{span}\{v_{i-1}, v_i\}, \quad i = 2, \ldots, n.$$

9. (15 %) Show that for any real numbers a, b, c, d,

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = 0.$$

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