編號：月 45 系所：數學系應用數學 科目：線性代數

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(1) One of the most often encountered determinants is the *Vandermonde determinant*, i.e., the determinant of the Vandermonde matrix

$$V(x_1, x_2, \ldots, x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}.$$

(a) [14%] Verify that $V(x_1, x_2, \ldots, x_n) = \prod_{i>j}(x_i - x_j)$.

(b) [14%] Compute

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_2x_3\cdots x_n \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_1x_3\cdots x_n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_1x_2\cdots x_{n-1} \end{vmatrix}$$

(2) [12%] Let A_1 and A_2 be $m \times n$ matrices, and let V_1 and V_2 be the spaces spanned by the rows of A_1 and A_2, respectively; let W_1 and W_2 be the spaces spanned by the columns of A_1 and A_2, respectively. Prove that the following conditions are equivalent:

(a) $\text{rank}(A_1 + A_2) = \text{rank}A_1 + \text{rank}A_2$;

(b) $V_1 \cap V_2 = 0$;

(c) $W_1 \cap W_2 = 0$.

(3) [12%] Consider $\mathbb{C}$ as a vector space over $\mathbf{R}$. Let A be a linear map of $\mathbb{C}$ into itself given by $Az = az + b\bar{z}$, where $a, b \in \mathbb{C}$. Prove that this map is not invertible if and only if

$$|a| = |b|.$$

(4) [12%] A matrix A is **Hermitian** if $A^* = A$, where $A^* = \overline{A}^T$ is obtained from A by complex conjugation of its elements and transposition. A Hermitian matrix A is called **positive** (resp. nonnegative) **definite** if $x^*Ax > 0$ (resp. $x^*Ax \geq 0$) for any nonzero vector x. Let A and B be Hermitian matrices. We will write that $A > B$ (resp. $A \geq B$) if $A - B$ is a positive (resp. nonnegative) definite matrix. The inequality $A > 0$ means that A is positive definite.

Let A be a Hermitian matrix. If $A > 0$, show that $A + A^{-1} \geq 2I$.

（背面仍有題目，請繼續作答）

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(5) Let V be a vector space over the field of real numbers. A **bilinear form** on V is a function $f: V \times V \to \mathbb{R}$ which is linear in either of its variables when the other variable is fixed. A bilinear form f on V is called **skew-symmetric** if $f(u,v) = -f(v,u)$ for all vectors u, v in V. If $f(u,v) = 0$ for all v in V implies that $u = 0$, then the bilinear form f is called **non-degenerate**. Now suppose f is a non-degenerate skew-symmetric bilinear form on V.

(a) [12%] Show that there are linearly independent vectors x, y in V such that $f(x,y) = 1$.

(b) [12%] Suppose x, y are vectors in V such that $f(x,y) = 1$. Let W be the two-dimensional subspace spanned by x and y. Let $W^{\perp}$ be the set of all vectors u in V such that $f(u,v) = 0$ for every v in the subspace W. Show that $V = W \oplus W^{\perp}$.

(c) [12%] Suppose V is finite dimensional. Show that there exists a basis $\{x_1, y_1, \dots, x_n, y_n\}$ for V such that

- $f(x_j, y_j) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$
- $f(x_i, x_j) = f(y_i, y_j) = 0.$

In particular, the dimension of V is even.