1. Let $M_2(\mathbb{R})$ be the set of all 2×2 real matrices and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Define a linear map $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$T(B) = AB - BA \quad \text{for any } B \in M_2(\mathbb{R}).$$

(a) (5%) Find $\ker T$.

(b) (5%) Find $\operatorname{Im} T$.

2. Let $X = C([a, b])$ be the set of all continuous real valued functions defined on the interval $[a, b]$. Let $\varphi : X \to X$ be defined by

$$(\varphi(f))(t) = \int_a^t f(x)dx.$$

(a) (5%) Show that φ is a linear transformation.

(b) (5%) Find $\ker \varphi$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by

$$T(x, y, z) = (x + y, x - y, x + y + z) \quad \text{for any } x, y, z \in \mathbb{R}.$$

(a) (5%) Show that $B = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of $\mathbb{R}^3$.

(b) (5%) What is the matrix of T with respect to the basis B?

4. Let $A = \begin{pmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{pmatrix}$.

(a) (5%) Find the characteristic polynomial of A.

(b) (5%) Find the eigenvalue(s) of A.

(c) (5%) Determine the eigenspace(s) for A.

(d) (5%) Find the Jordan normal form of A.

(背面仍有題目,請繼續作答)

5. Let X_1 and X_2 be subspaces of a finite dimensional vector space V. Show that

(a) (5 %) $X_1 \cap X_2$ is also a subspace of V

(b) (5 %) $\dim(X_1 + X_2) = \dim X_1 + \dim X_2 - \dim(X_1 \cap X_2)$.

6. A linear transformation T of a real inner product space V is said to be skew self-adjoint if

$$\langle Tx, y\rangle = -\langle x, Ty\rangle \quad \text{for any } x, y \in V.$$

Prove that

(a) (10%) T is skew self-adjoint if and only if the matrix A of T with respect to an orthonormal basis is antisymmetric, i.e., $A^t = -A$, where A^t is the transpose of A.

(b) (10%) Suppose that T is skew self-adjoint and $T(W) \subset W$. Show that $T(W^{\perp}) \subset W^{\perp}$, where $W^{\perp} = \{v \in V | \langle v, w\rangle = 0 \text{ for all } w \in W\}$ is the orthogonal complement of W in V.

7. Let V be a complex finite dimensional vector space and ϕ and ψ two diagonalizable endomorphisms such that $\phi \circ \psi = \psi \circ \phi$. Note that an endomorphism T of V is called diagonalizable if there exists a basis B of V such that the matrix of T with respect to B is a diagonal matrix.

(a) (10%) Let λ be an eigenvalue of ϕ and let $E_\lambda = \{v \in V \,|\, \phi v = \lambda v\}$ be the eigenspace of ϕ of eigenvalue λ. Show that $\psi(E_\lambda) \subset E_\lambda$.

(b) (10%) Show that there is a basis B of V such that the matrices of ϕ and ψ with respect to B are both diagonal.

The End