1. The sales of a convenience store on a randomly selected day are X thousand dollars, where X is a random variable with a distribution function of the following form:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}x^2, & 0 \le x < 1 \\ k(4x - x^2), & 1 \le x < 2 \\ 1, & x > 2. \end{cases}$$

Suppose that this convenience store's total sales on any given day are less than \$ 2000.

(a) Find the value of k. (5%)

(b) Let A and B be the events that tomorrow the store's total sales are between 500 and 1500 dollars, and over 1000 dollars, respectively. Find $P(A)$ and $P(B)$. (10%)

(c) Are A and B independent events? (5%)

2. Let (X, Y) be a continuous random vector with the probability density function

$$f(x, y) = \begin{cases} 4x(1-y), & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $E(X^j Y^k)$, $j, k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. (10%)

(b) Find $\text{Var}(X - Y)$ and $\rho(X, Y)$ (the correlation coefficient of X and Y). (10%)

3. Suppose that $X, Y \in L^2$.

(a) Show that

$$\text{Var}(X) = \text{Var}[E(X|Y)] + E[\text{Var}(X|Y)],$$

where $\text{Var}(X|y) = E\{[X - E(X|y)]^2 | y\}$. (10%)

(b) For each $\theta \in [0, 2\pi]$, define

$$X_\theta = X\cos\theta - Y\sin\theta$$

$$Y_\theta = X\sin\theta + Y\cos\theta.$$

Show that there is at least one value of θ for which X_θ and Y_θ are uncorrelated. (10%)

4. Let $f(x, y)$ be the joint probability density function of continuous random variables X and Y; f is called a bivariate normal probability density function if

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} q(x, y)\right], \quad (x, y) \in \mathbb{R}^2,$$

where ρ is the correlation coefficient of X and Y and

$$q(x, y) = \left(\frac{x - \mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x - \mu_X}{\sigma_X}\right)\left(\frac{y - \mu_Y}{\sigma_Y}\right) + \left(\frac{y - \mu_Y}{\sigma_Y}\right)^2$$

$(\mu_X, \mu_Y \in \mathbb{R}, \sigma_X, \sigma_Y > 0, -1 < \rho < 1)$.

(a) Find the conditional distribution of Y, given $X = x$ $(\in \mathbb{R})$. (10%)

(b) For what values of α is the variance of $\alpha X + Y$ minimum? (5%)

(c) Show that if $\sigma_X = \sigma_Y$, then $X + Y$ and $X - Y$ are independent random variables. (5%)

5. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequcnec of i.i.d. r.v.'s with common probability density function

$$f(x) = \begin{cases} e^{-(x-\theta)}, & \text{if } x \ge \theta \\ 0, & \text{otherwise} \end{cases}$$

Write $\overline{X}_n = \sum_{i=1}^{n} X_i/n$, $X_{(1)} = \min\{X_1, \ldots, X_n\}$.

(a) Show that $\overline{X}_n \xrightarrow{p} 1 + \theta$. (10%)

(b) Show that $X_{(1)} \xrightarrow{p} \theta$. (10%)