Ordinary Differential Equation

(1) Solve the following differential equations: (20%)

(a) $\frac{dy}{dx} = \frac{2x+xy}{y^2+1}$

(b) $\frac{dy}{dx} = \frac{x+2y+2}{-2x+y}$.

(c) $x^2y'' = y'(3x-2y')$.

(d) $y''-y'-2y = 4x^2$.

(2) Study the asymptotic behabior of the differential equation (10%)

$$y'(x) = (y-1)(y-2)(y-3) \qquad y(0) = c \in \mathbf{R}$$

as $x \to \infty$ without solving the equation. You need to separate the range of c. Can you sketch the integral curve?

(3) Use the method of variation of parameters to show that (20%)

$$y(x) = c_1 \cos x + c_2 \sin x + \int_0^x f(\xi)\sin(x-\xi)d\xi$$

is a general solution to the 2nd order differential equation

$$y'' + y = f(x)$$

where $f(x)$ is a continuous function on $(-\infty, \infty)$.

(4) Find the solution to the initial value problem (20%)

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2e^{5t} \\ 2e^{2t} \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(5) Apply the Laplace transform to solve the initial value problem (15%)

$$y'' + 2ty' - 4y = 1, \qquad y(0) = y'(0) = 0$$

(6) The Bessel function is defined by (15%)

$$J_\nu(x) \equiv \frac{1}{\pi}\int_0^\pi \cos(\nu t - x\sin t)dt \qquad -\infty < x < \infty, \quad \nu \in \mathbf{R}$$

where f is a continuous and differentiable function. Show that when ν is a interger then it satisfies the Bessel equation

$$x^2 J_\nu''(x) + xJ_\nu'(x) + (x^2-\nu^2)J_\nu(x) = 0$$

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