1. Let T be a linear transformation form the vector space V into the vector space W and $v_1, v_2, \ldots, v_n$ be a basis for V. Prove that
   (i) T is onto if and only if $T(v_1), T(v_2), \ldots, T(v_n)$ span W. (6%)
   (ii) T is one-to-one if and only if $T(v_1), T(v_2), \ldots, T(v_n)$ is linearly independent. (8%)

2. Let $A, B \in M_{n\times n}(\mathbb{R})$, show that
   (i) if A and B are upper triangular, than AB is alos upper triangular; (5%)
   (ii) rank $AB =$ rank BA is not always true; (5%)
   (iii) rank $A \leq 1$ if and only if there exist $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ such that $A = x^t y$. (10%)

3. Suppose $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+d & a+b+c+d \\ 0 & -a-d \end{bmatrix}$.
   (i) Find ker (T), Im (T), nullity (T) and rank (T). (8%)
   (ii) Find the characteristic polynomial and the minimal polynomial of T. (10%)
   (iii) Is T diagonalizable? Why? (4%)

4. Let $A = \begin{bmatrix} 6 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 5 & 1 \\ \sqrt{2} & 1 & 5 \end{bmatrix}$.
   (i) Find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal. (10%)
   (ii) Find a positive-definite matrix B such that $B^2 = A$. (5%)
   (iii) Evaluate $\lim_{n\to\infty} (\frac{1}{8}A)^n$. (5%)

5. Let V be a fine-dimensional inner product space and $T : V \to V$ be linear. Prove the following statements:
   (i) If the minimal polynomial of T is $m(t) = p(t)q(t)$ where $p(t)$ and $q(t)$ are relative prime polynomials, then $V = \ker(p(T)) \oplus \ker(q(T))$. (10%)
   (ii) If T is idempotent (i.e. $T^2 = T$), then $V = \ker(T) \oplus \mathrm{Im}(T)$. (4%)
   (iii) If T is idempotent, then T is self-adjoint if and only if $\ker(T) \perp \mathrm{Im}(T)$. (10%)