1. 直接寫下以下各常微分方程式之解 $y(x)$. (10%)

   (a) $\dfrac{d}{dx}y = y,\ y(0) = 1$.
       (b) $\dfrac{d^2}{dx^2}y = -y,\ y(0) = 1,\ y'(0) = 0$.

   (c) $\dfrac{d^2}{dx^2}y = y,\ y(0) = 0,\ y'(0) = 2$.
       (d) $\dfrac{d}{dx}y = x,\ y(0) = 0$.

2. 求以下各常微分方程式之所有解. (20%)

   (a) $\dfrac{d^2}{dx^2}y = -4y,\ y(0) = 1$.
       (b) $\dfrac{d}{dx}y = y + x$.

   (c) $\dfrac{dy}{dx} = xy$.
       (d) $2xy\dfrac{d}{dx}y = x^2 + y^2$.

3. 方陣 $A = \begin{bmatrix} -\dfrac{5}{2} & \dfrac{3}{2} \\ \dfrac{3}{2} & -\dfrac{5}{2} \end{bmatrix}$ 且 $A = Q\Lambda Q^T$,其中 $Q = \dfrac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}$

   (a) 解一階聯立方程式 (10%)

$$\frac{d}{dx}Y(x) = AY + x\begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   (b) 解二階聯立方程式 (10%)

$$\frac{d^2}{dx^2}Y(x) = AY,\quad Y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\quad \frac{d}{dx}Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

4. $\dfrac{d}{dx^2}y + \lambda\dfrac{d}{dx}y + Py = 0,\ y(0) = 1,\ \dfrac{d}{dx}y(0) = 0$,其中 $\lambda, P > 0$,求令 $y(x)$

   在區間 $(0, \infty)$ 不為零之條件(λ 與 P 之關係). (10%)

5. $\dfrac{d^2}{dx^2}y + \omega^2 y = \sin kx,\ y(0) = 0,\ \dfrac{d}{dx}y(0) = 0$. (10%)

   (a) 求以上初始值問題之解. (設 $\omega \neq k$)

   (b) 求 k 趨近於 ω 時之解. ($\lim\limits_{k \to \omega} y(x)$)

6. 求以下初始值問題之 power series 解. (10%)

$$(1 - x^2)\frac{d^2}{dx^2}y + 2x\frac{d}{dx}y + 6y = 0,\quad y(0) = 0,\quad \frac{d}{dx}y(0) = 1.$$

7. 函數 $u(x, t)$ 滿足 wave equation

$$\frac{\partial^2}{\partial x^2}u(x, t) = \frac{\partial^2}{\partial t^2}u(x, t),$$

   邊界條件 $u(0, t) = 0$ 與 $u(2\pi, t) = 0$,初始條件 $u(x, 0) = \sin x + \sin 2x$.
   求 $u(x, t)$ 之解. (20%)