1. Let X and Y be random variables and let A be an event. Prove that the function

$$Z(w) = \begin{cases} X(w), & \text{if } w \in A \\ Y(w), & \text{if } w \in A^c \end{cases}$$

is a random variable. (7%)

2. Let X and Y be i.i.d. with continuous distribution function F.
Find the probabilities $P(X = Y)$ and $P(X < Y)$. (8%)

3. Let X_1, X_2 be independently distributed as $N(\mu_i, \sigma_i^2)$, $\sigma_i > 0$, $i = 1, 2$, and let

$$\begin{cases} Z_1 = X_1 \cos\theta + X_2 \sin\theta \\ Z_2 = X_2 \cos\theta - X_1 \sin\theta. \end{cases}$$

Find the correlation coefficient between Z_1 and Z_2, and show that

$$0 \le \rho^2 \le \left(\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^2,$$

where ρ denotes the correlation coefficient of Z_1 and Z_2. (10%)

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of events such that $\lim_{n\to+\infty} A_n = A\ (\in \mathcal{F})$,
   (a) Show that $\lim_{n\to+\infty} P(A_n) = P(A)$. (15%)
   (b) Prove that if $\sum_{n=1}^{+\infty} P(A_n) < +\infty$, then $P(\bigcap_{k=1}^{+\infty} \bigcup_{n=k}^{+\infty} A_n) = 0$. (10%)

5. Let $X_1, \ldots, X_n$ be independently distributed as $N(\mu, \sigma^2)$, $\sigma > 0$.
   (a) Prove that $\overline{X}$ and $\underset{\sim}{Y} = (X_1 - \overline{X}, \ldots, X_n - \overline{X})'$ are independent. (10%)
   (b) Prove that $\frac{nS^2}{\sigma^2}$ is distributed as χ^2_{n-1}, where $S^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j - \overline{X})^2$. (10%)

6. (a) Let X, Y be random variables on the probability space $(\Omega, \mathcal{F}, P)$. Assume that $p, q > 1 \ni \frac{1}{p} + \frac{1}{q} = 1$ and $E|X|^p < +\infty$, $E|Y|^q < +\infty$. Prove that $E|XY| \le (E|X|^p)^{\frac{1}{p}}(E|Y|^q)^{\frac{1}{q}}$. (15%)
   (b) Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables and let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. Prove that if $X_n \xrightarrow{\text{q.m.}} X$, then (15%)
      (i) $EX_n \underset{n\to+\infty}{\longrightarrow} EX$,
      (ii) $EX_n^2 \underset{n\to+\infty}{\longrightarrow} EX^2$, and hence $\text{Var}(X_n) \underset{n\to+\infty}{\longrightarrow} \text{Var}(X)$.

   (Note. $X_n \xrightarrow{\text{q.m.}} X$ means that $\{X_n\}_{n\in\mathbb{N}}$ converges to X in quadratic mean as $n \to +\infty$.)

706002A