Work out all problems with details to get full credits.

(1) (a) Let V be $\mathbb{R}^2$, with the standard inner product. If U is an unitary operator on V, show that the matrix of U in the standard ordered basis $\{e_1 = (1,0), e_2 = (0,1)\}$ is either (10%)

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

for some real θ, $0 \le \theta \le 2\pi$.

(b) Let ϕ be a fixed real number, and let $B = \{\alpha_1, \alpha_2\}$ be the orthonormal basis obtained by rotating $\{e_1, e_2\}$ through the angle ϕ. Let θ be another real number. Find the matrix $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ in the ordered basis B. (10%)

(2) (a) Let A be a square matrix satisfying $A^2 = A$. Find the distinct eigenvalues of A. (10%)

(b) Let A be a square matrix of rank k. Show that the number of distinct nonzero eigenvalues of A is less than or equal to k. (10%)

(c) Consider the $n \times n$ matrix (10%)

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Find the eigenvalues of A. (Hint. Try some eigenvectors of A. What is the rank of A?)

(3) Let T be the linear operator on $\mathbb{R}^2$, the matrix of which in the standard ordered basis is $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$. Let W_1 be the subspace of $\mathbb{R}^2$ spanned by the vector $e_1 = (1,0)$.

(a) Show that W_1 is invariant under T. (5%)

(b) Prove that there is no subspace W_2 which is invariant under T and which is complementary to W_1 (i.e., no subspace W_2 satisfying $\mathbb{R}^2 = W_1 \oplus W_2$.) (5%)

Let $P_2(\mathbb{R})$ be the vector space of all polynomial functions p from $\mathbb{R}$ to $\mathbb{R}$ which has degree 2 or less (so $p(x) = c_0 + c_1 x + c_2 x^2$ for some $c_i \in \mathbb{R}$). Also, denote by $M_n(\mathbb{R})$ the set of all $n \times n$ matrices over $\mathbb{R}$.

(4) Define three linear functionals on $P_2(\mathbb{R})$ by (10%)

$$f_1(p) = \int_0^1 p(x)\,dx, \quad f_2(x) = \int_0^2 p(x)\,dx, \quad f_3(x) = \int_0^{-1} p(x)\,dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis of the dual space $P_2(\mathbb{R})^*$. (Hint. Find the dual basis of it in $P_2(\mathbb{R})$.)

(5) Compute the minimal polynomial for each of the following linear operators.

(a) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, where $T(p) = p' + 2p$; (5%)

(b) $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, where $T(A) = A^t$, the transpose of A. (5%)

(6) Let V be a finite dimensional vector space over $\mathbb{C}$, and A and B two commuting linear transformations of V. Show that there is at least one common eigenvector for A and B. (10%)

(背面仍有題目,請繼續作答)

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(7) Let V be the linear subspace spanned by $(1,0,2,3)$, $(2,3,0,1)$, $(3,1,2,0) \in \mathbb{R}^4$. Find an orthonormal basis of V. (10%)