(1) Suppose that you are distributing three balls, numbered a, b and c, into three boxes, numbered 1, 2, and 3. Each box may contain more than one ball. Assume that each arrangement is equally likely. Let A be the event "one of the boxes is occupied by more than one ball", while the event B means "the first box is occupied at least by ball a". Let $\mathbf{1}_A$ be the indicator function of A, that is,

$$\mathbf{1}_A = \begin{cases} 1, & \text{if event } A \text{ happened;} \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, we can define for $\mathbf{1}_B$. In this problem we consider two random variables X and Y where $X = \min\{\mathbf{1}_A, \mathbf{1}_B\}$ and $Y = \max\{\mathbf{1}_A, \mathbf{1}_B\}$.

(i) How many possible outcomes are there in the sample space? 10%

(ii) Are X and Y independent? Why? 10%

(iii) Compute the conditional expectation $E(2X - Y|Y = 1)$. 10%

(2) Let X_0 and X_1 represent the lifetimes of two machines, machine 0 and machine 1. Suppose they are independent exponentially distributed with respective parameters λ_0 and λ_1 so that

$$P\{X_i > t\} = e^{-\lambda_i t}, \text{ for } t \geq 0, i = 0, 1.$$

Define two random variables N and U as follows:

$$N = \begin{cases} 1, & \text{if } X_1 < X_0; \\ 0, & \text{otherwise.} \end{cases}$$

$$U = \min\{X_0, X_1\}.$$

That is, $N = 1$ if the machine 1 has a shorter lifetime than the machine 0, while U records the time that the machine is dead, whichever comes first.

(i) Compute $E(X_1 X_2)$. 10%

(ii) Use the joint density function of X_0 and X_1 to compute $P\{N = 0, U > t\}$, the probability that the machine 0 dies first and it does so after time t. 10%

(iii) Notice that U is a positive random variable. What are the marginal density functions of N and U? 10%

(3) A roulette wheel has 38 slots – 18 red, 18 black, and 2 green ones. If a gambler bets \$1 on red coming up, he wins \$1 with probability 18/38 and loses \$1 with probability 20/38. Let $X_1, X_2, \ldots$ be the outcomes of successive bets, taking values only on 1 (win) or -1 (lose), and are assumed to be *i.i.d.* random variables. Denote S_n to be the amount he has won/lost after n bets. Clearly, $E(X_1) = -\$\frac{1}{19}$ and $E(S_n) = -\$\frac{n}{19}$.

(i) What conclusion can be made by *the weak law of large numbers*(WLLN)? 10%

(ii) By the WLLN, is it true that, almost every gambler will definitely loss money and will never win back again if he has played long enough(i.e. $S_n < 0$ for $n > N$)? Why? 10%

(iii) For $n = 100$, the gambler is expected to have lost \$5.26 on the average. However, there is still some chance that the gambler's money remained nonnegative at $n = 100$. Find out this probability. 10%

(4) Let ξ have a Poisson distribution with mean λ. Compute the generating function $E(s^{\xi})$, and then use it to find the variance of ξ. 10%

Table of the Normal Distribution

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

To illustrate the use of the table: $\Phi(0.36) = 0.6406$, $\Phi(1.34) = 0.9099$

	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9986	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990