(1) (60%) Find the general solution of the following equation or the system of equations.

(a) $(1 + x^2)\frac{dy}{dx} + 4xy = (1 + x^2)^{-2}$.

(b) $(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2\cos x)dy = 0$.

(c) $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$.

(d) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2 + 4xe^{2x}$.

(e) $\frac{d^2y}{dx^2} + 9y = 9\sec^2 3x$, for $0 < x < \pi/6$.

(f) $x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 0$.

(g) $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = 0$, for $x > 0$.

Hint: you may use the fact that $y_1(x) = \sin x^2$ is a solution of the equation.

(h) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$.

Hint: You may set $v = \frac{dy}{dx}$.

(i)
$$\frac{dY}{dx} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} Y$$

(j)
$$\frac{dY}{dx} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} Y$$

(2) (10%) Find the general solution of the differential equation
$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0,$$
by means of a power series about $x_0 = 1$.

(3) (10%) Solve the following Riccati equation
$$\frac{dy}{dx} = 1 + x^2 - 2xy + y^2.$$

Hint: you may use the fact that $y_1(x) = x$ is a solution of the equation.

(背面仍有題目,請繼續作答)

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(4) (10%) Show that if

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = f(x,y)$$

is **homogeneous**, i.e. f depends only on the ratio y/x, then it has

$$\mu(x,y) = \frac{1}{xM(x,y) + yN(x,y)}$$

as an integrating factor.

(5) (10%) Show that the system

$$\frac{dx}{dt} = x + y + x^3 - y^2, \quad \frac{dy}{dt} = -x + 2y + x^2y + \frac{y^3}{3}$$

has no periodic solutions other than constant solutions.