(1) Suppose $f(x) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is an even function vanishing outside a disk of radius r centered at $\mathbf{0} = (0,0)$, that is, $f(x) = f(-x)$ and $f(x)$ is non-zero only for x in a r-disk. If $\mathbb{Z}^2 = \{(z_1, z_2) | z_1, z_2 \in \mathbb{Z}\}$ is a two-dimensional lattice, and define

$$g(k) = \sum_{x \in \mathbb{Z}^2} e^{<k,x>} f(x), \; k = (k_1, k_2) \in \mathbb{R}^2,$$

where $< k, x >$ is the usual inner product.

(i) Show that $g(k)$ is also an even function. 10%

(ii) Compute $\nabla g(\mathbf{0})$. 10%

(iii) Show that 10%

$$g(k) = \sum_{x \in \mathbb{Z}^2} f(x) + \int_0^1 (1-t) \frac{d^2}{dt^2} g(tk) dt.$$

(Hint: integration by parts)

(iv) Prove the following estimation: 10%

$$|g(k) - g(\mathbf{0})| \leq \frac{1}{2} \sup_{0 \leq t \leq 1} \left| \sum_{i,j=1}^{2} \partial^2_{k_i k_j} g(tk) k_i k_j \right|$$

(2) Let d stand for a positive integer throughout the rest of this exam sheet.

(i) If $\sum a_n$ is a series with positive terms, state and prove the *Ratio Test* for the convergence of $\sum a_n$. 10%

(ii) Suppose $a_n = (1 + \frac{1}{\sqrt{d}})^{n-2} (\frac{1}{\sqrt{d}})^{n-2} (2n-1)^2$. For what values of d does the infinite series $\sum_{n=2}^{\infty} a_n$ converge? 10%

(iii) Now let $b_n = (1 + \frac{1}{\sqrt{d}})^{n-2} (\frac{1}{\sqrt{d}})^{n-1} (2n-1)^2$. Show that there exists a constant K_0 such that $\sum_{n=3}^{\infty} b_n \leq K_0 d^{-1}$ for all sufficiently large d. In other words, $\sum_{n=3}^{\infty} b_n = O(d^{-1})$. 10%

(3) Let $D(k) = \frac{1}{d} \sum_{j=1}^{d} \cos k_j$, where $k = (k_1, k_2, \ldots, k_d)$ with $k_j \in [-\pi, \pi]$ for all $j = 1, 2, \ldots, d$. Define $k^2 = \sum_{j=1}^{d} k_j^2$.

(i) Write down the Taylor series with remainder for $\cos x$ around $x = 0$ and show that this Taylor series converges for all $x \in \mathbb{R}$. 10%

(ii) Explain how the *Mean Value Theorem* is a special case of *Taylor's Theorem*. 10%

(iii) Notice that the Taylor series of $\cos x$ is an alternating series. Show that 10%

$$\frac{2k^2}{\pi^2 d} \leq 1 - D(k) \leq \frac{k^2}{2d}.$$