

Differential Equations

1. (14%) Solve

$$(3xy^2 - 6y) dx + (2x^2y - 3x) dy = 0.$$

2. (14%) Find a solution to the equation

$$y''(x) + y(x) = \delta(x),$$

where $\delta(x)$ is the Dirac delta function.

3. (14%) Find the matrix $X(t)$ that satisfies

$$X'(t) = \begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} X(t), \quad X(0) = I,$$

where I is the identity matrix.

4. (14%) Determine conditions on f that guarantee that the problem

$$y''(x) + 4y'(x) + 5y(x) = f(x), \quad y(0) = y(\pi) = 0$$

has a solution.

5. (15%) Find all the real eigenvalues and eigenfunctions for

$$x^2y'' + xy' + \lambda y = 0, \quad y'(1) = 0, \quad y(e) = 0.$$

6. (15%) Consider the system

$$\begin{aligned} x' &= x - y, \\ y' &= 3x + y - y^2. \end{aligned}$$

Find all the critical points and discuss the stability (asymptotically stable, stable, or unstable) of each critical point.

7. (14%) Find a solution to the boundary value problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & 0 < x < \pi, & \quad 0 < y < 1, \\ u_x(0, y) &= u_x(\pi, y) = 0, & 0 < y < 1, \\ u(x, 1) &= 0, & 0 \leq x \leq \pi, \\ u(x, 0) &= 3 \cos(2x) + 2 \cos(3x), & 0 \leq x \leq \pi. \end{aligned}$$